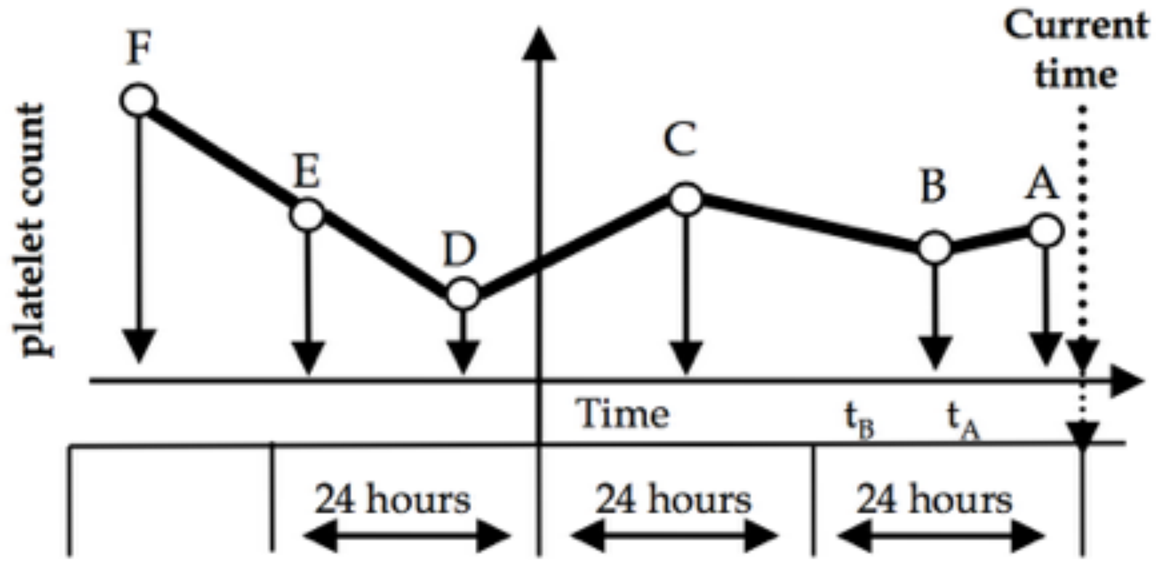
10 YEARS ROAD TO SQUEAK

SequeL, Inria Lille - Nord Europe

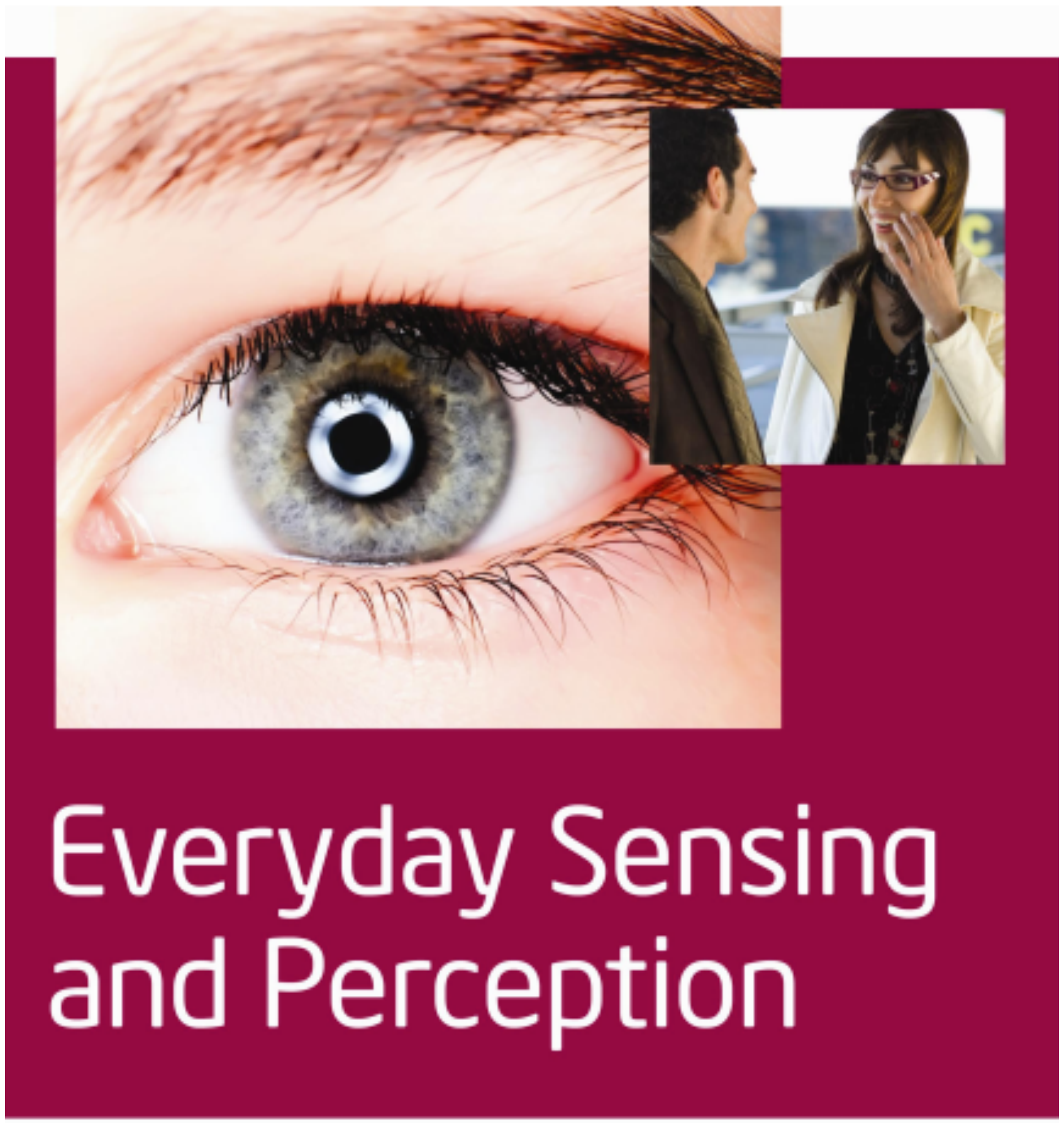
10 YEARS ROAD TO SQUEAK AND QUADRATIC BARRIER

ONLINE GRAPH-BASED ANOMALY DETECTION

- medical data
- graph on patient states
- labels are the medical action
- goal: online detection of anomalous data



EVERYDAY SENSING AND PERCEPTION



EVERYDAY SENSING AND PERCEPTION

informatics mathematics Inria

Intel Research Berkeley

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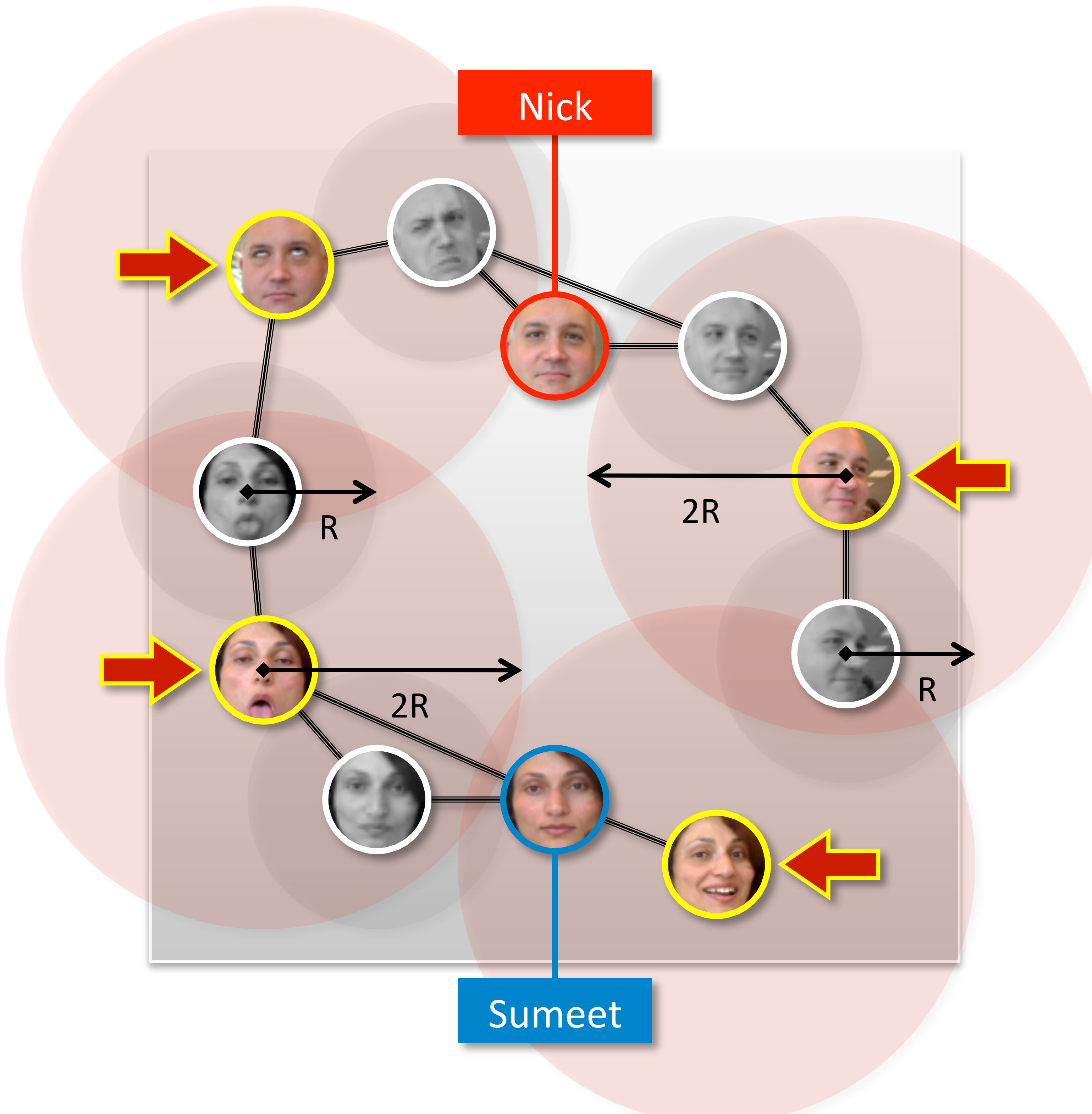
Online Semi-Supervised Learning and Face Recognition

This project focuses on real-time learning without explicit feedback. This work combines the ideas of semi-supervised learning on approximate graphs and online learning. In particular, we develop algorithms that iteratively build a graphical representation of the world and update it on-the-fly with observed examples (both labeled and unlabeled). We proved regret bounds of the solutions, demonstrated that the system can recognize faces in real-time even in a resource constraint environment and can take advantage of the manifold structure to outperform existing methods. The following videos show how online semi-supervised learning can be used to train a robust face recognizer of a person from just a single frontal image:

ONLINE K-CENTER CLUSTERING



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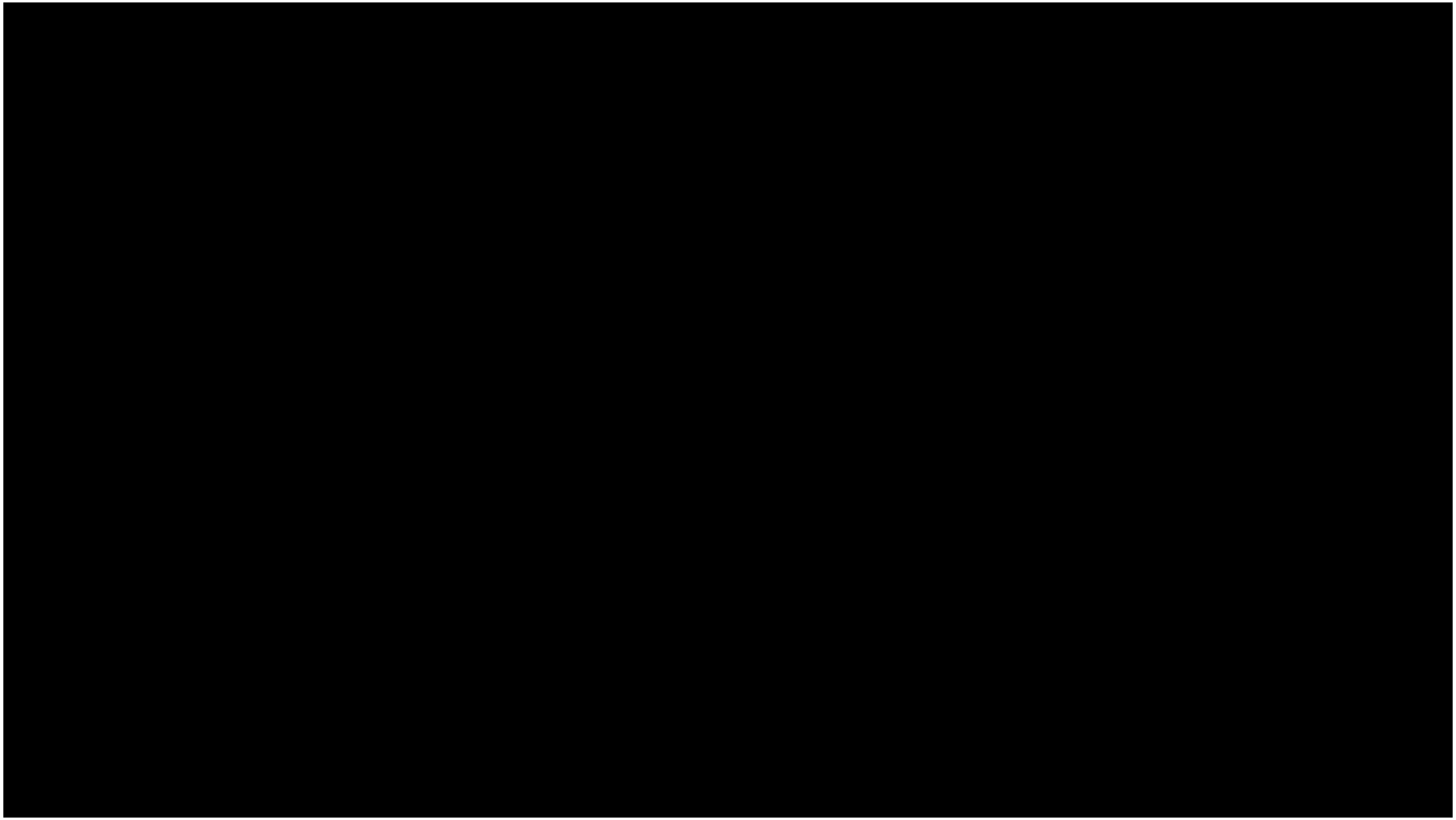
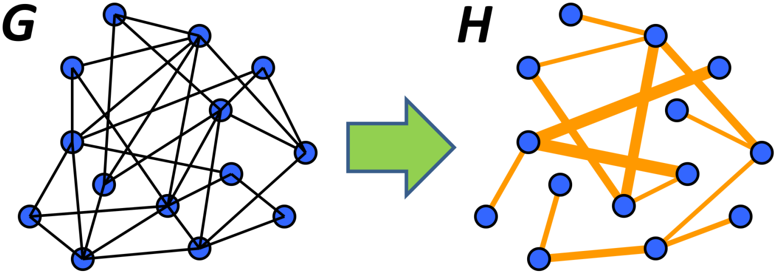
INTEL AD FOR THE ONLINE FACE RECO



Graph Sparsification

Goal: Get graph G and find sparse H



Graph Sparsification: What is sparse?

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in a tree — sure, in a dense graph perhaps not

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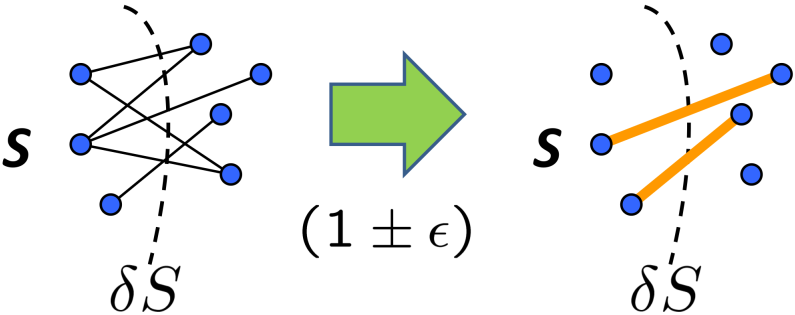

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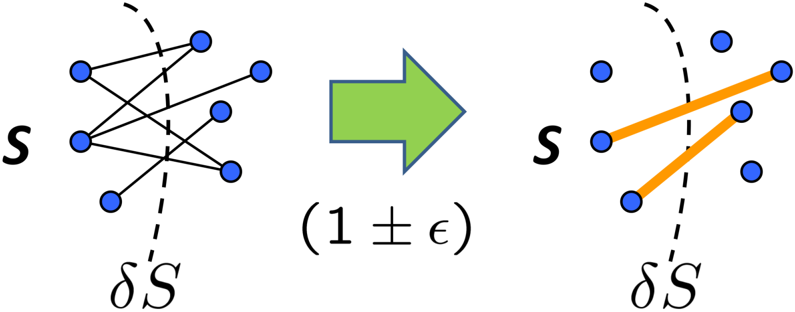

H approximates G well iff $\forall S \subset V$, sum of edges on δS remains

δS = edges leaving S

https://math.berkeley.edu/~nikhil/

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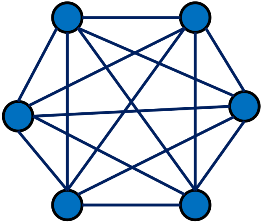
Is this always possible? Benczúr and Karger (1996): Yes!

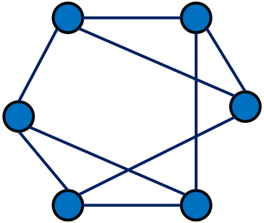
$\forall \varepsilon \; \exists \; (1+\varepsilon)$-cut similar $\widetilde{G}$ with $\mathcal{O}(n \log n / \varepsilon^2)$ edges s.t. $E_H \subseteq E$ and computable in $\mathcal{O}(m \log^3 n + m \log n / \varepsilon^2)$ time n nodes, m edges

Graph Sparsification: What is **good** sparse?

$G = K_n$



$H = d$-**regular** (random)



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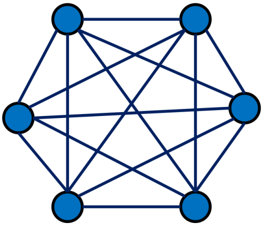

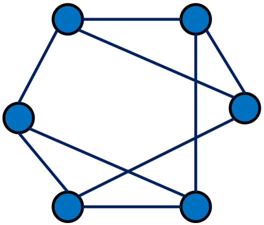

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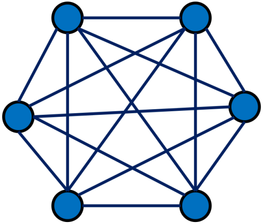

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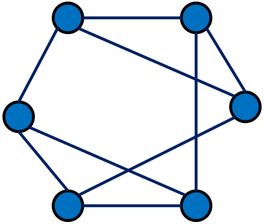

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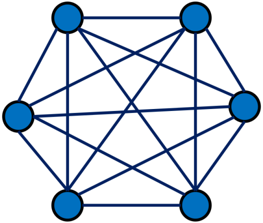

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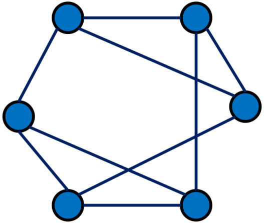

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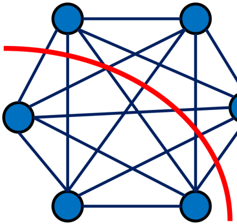

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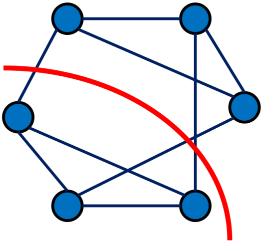

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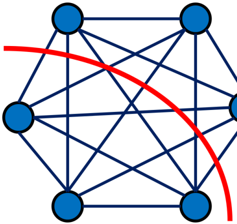

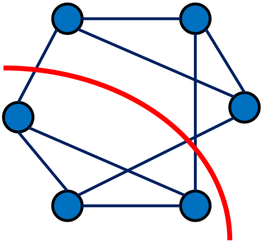

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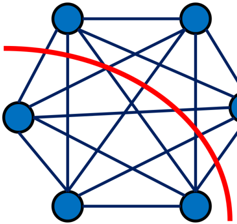

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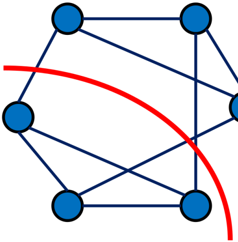

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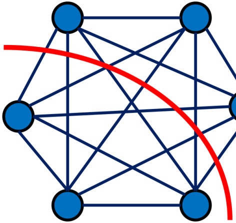

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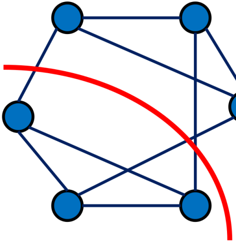

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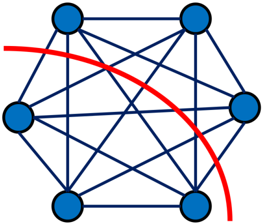

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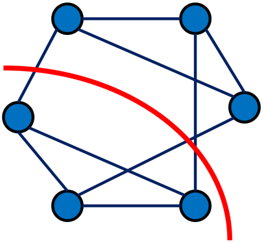

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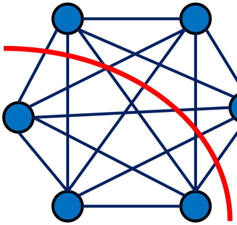

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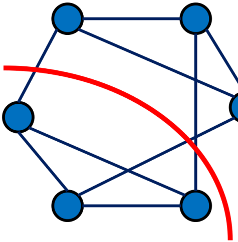

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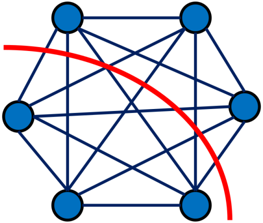

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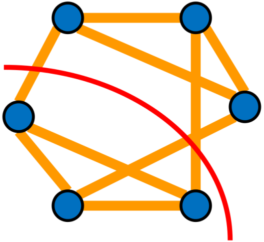

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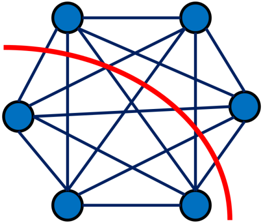

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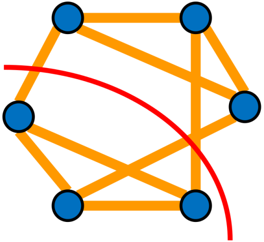

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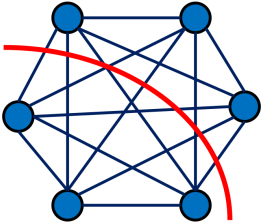

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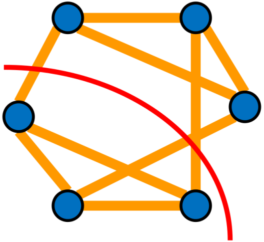

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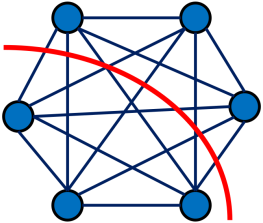

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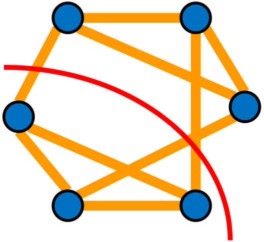

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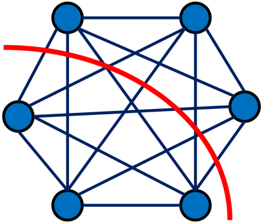

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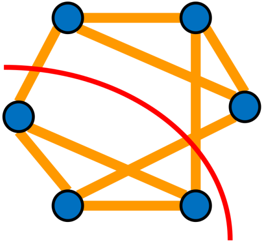

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Benczúr & Karger: Can find such H quickly for any G!

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Spectral sparsifiers are stronger!

Graph Sparsification: What is good sparse?

Recall if $\mathbf{f} \in \{0,1\}^n$ represents S then $\mathbf{f}^\mathsf{T}\mathbf{L}_G\mathbf{f} = \mathrm{cut}_G(S)$

$$(1-\varepsilon)\mathrm{cut}_H(S) \leq \mathrm{cut}_G(S) \leq (1+\varepsilon)\mathrm{cut}_H(S)$$

becomes

$$(1-\varepsilon)\mathbf{f}^\mathsf{T}\mathbf{L}_H\mathbf{f} \leq \mathbf{f}^\mathsf{T}\mathbf{L}_G\mathbf{f} \leq (1+\varepsilon)\mathbf{f}^\mathsf{T}\mathbf{L}_H\mathbf{f}$$

If we ask this only for $\mathbf{f} \in \{0,1\}^n \rightarrow (1+\varepsilon)$-cut similar combinatorial
Benczúr & Karger (1996)

If we ask this for all $\mathbf{f} \in \mathbb{R}^n \rightarrow (1+\varepsilon)$-**spectrally similar**
Spielman & Teng (2004)

Spectral sparsifiers are stronger!

but checking for spectral similarity is easier

Spectral Graph Sparsification

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Spectral Graph Sparsification

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Eigenvalues are approximated well!

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As a consequence, $\arg\min_{\mathbf{x}} \|\mathbf{L}_H\mathbf{x} - \mathbf{b}\| \approx \arg\min_{\mathbf{x}} \|\mathbf{L}_G\mathbf{x} - \mathbf{b}\|$

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Let us consider unweighted graphs: $w_{ij} \in \{0, 1\}$

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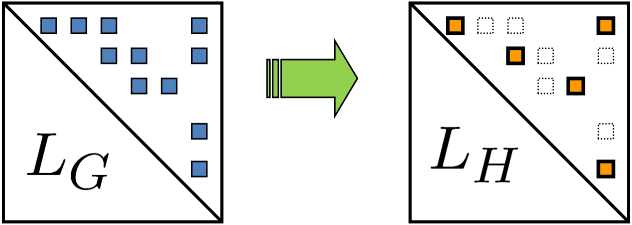

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How to get it?

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$$\text{Then } \textstyle\sum_{e \in E} s_e \mathbf{v}_e \mathbf{v}_e^\mathsf{T} \approx \mathbf{I} \iff \sum_{e \in E} s_e \mathbf{a}_e \mathbf{a}_e^\mathsf{T} \approx \mathbf{A}$$

multiplying by $\mathbf{A}^{1/2}$ on both sides

Spectral Graph Sparsification: Intuition

How does $\sum_{e \in E} \mathbf{v}_e \mathbf{v}_e^\mathsf{T} = \mathbf{I}$ look like geometrically?

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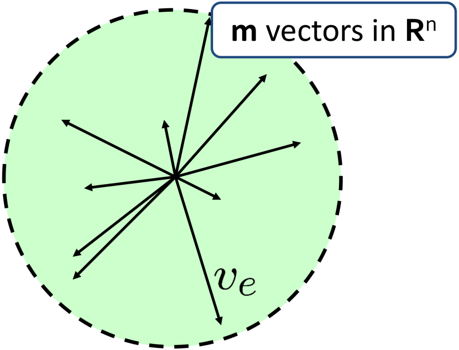

Decomposition of identity: $\forall \mathbf{u}$ (unit vector): $\sum_{e \in E} (\mathbf{u}^\mathsf{T} \mathbf{v}_e)^2 = 1$

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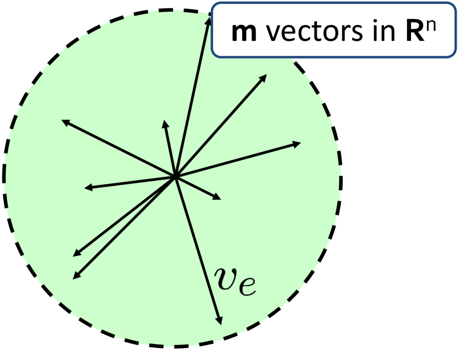

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moment ellipse is a sphere

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What are we doing by choosing H?

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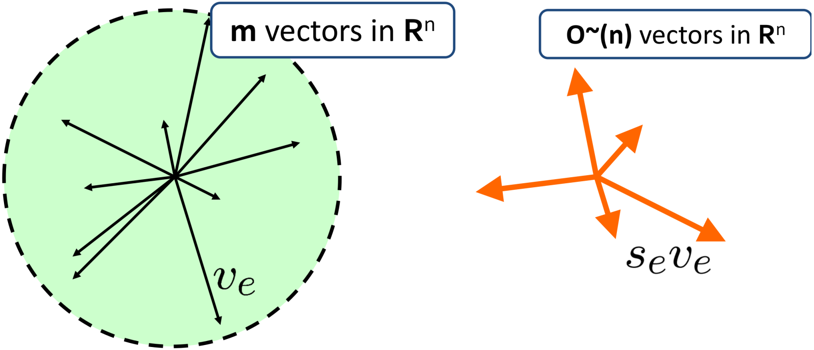

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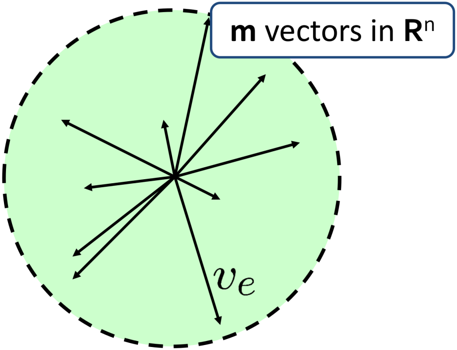

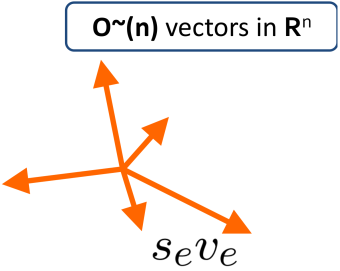

We take a subset of these $\mathbf{e}_e$s and scale them!

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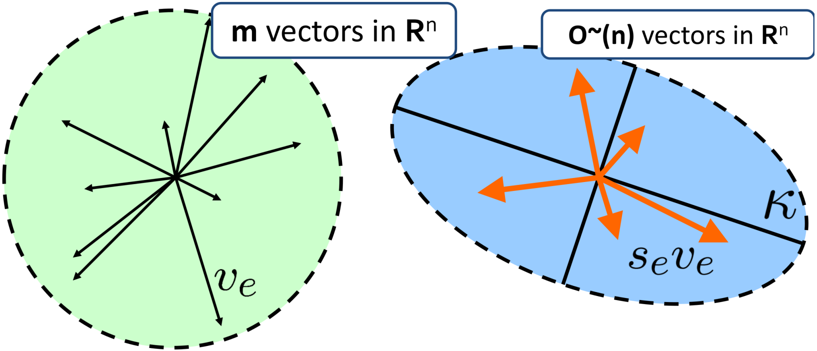

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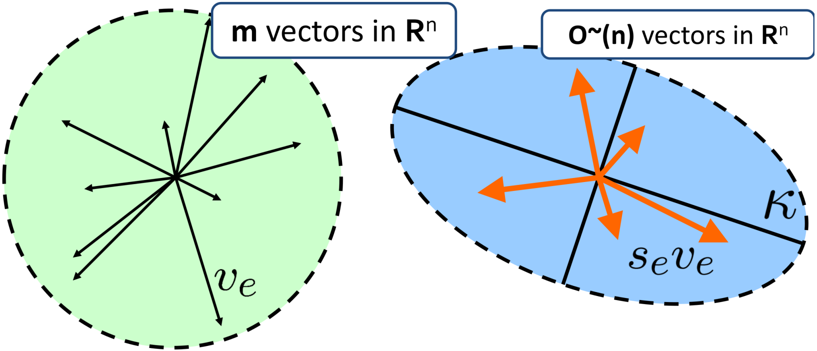

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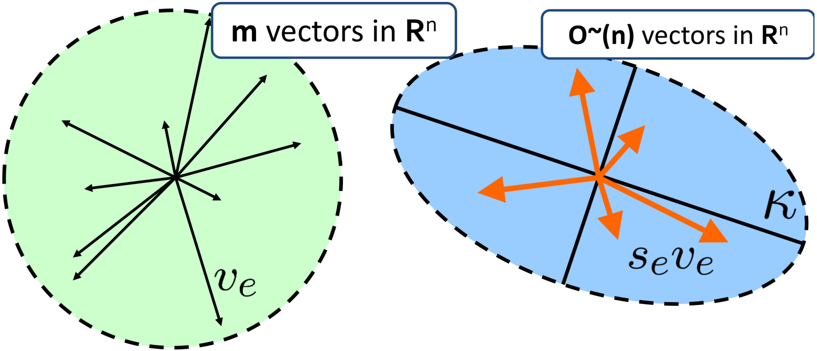

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the blue eigenvalues are between 1 and κ

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Example: What happens with K_n?

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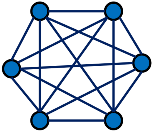
K_n graph



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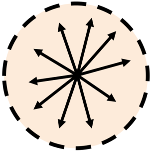

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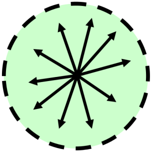

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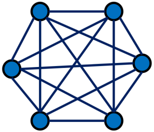

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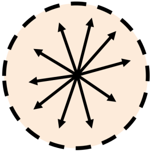

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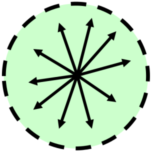

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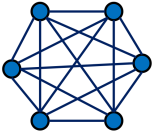

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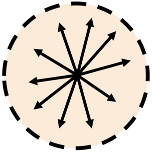

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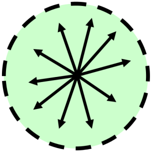

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rescaling $\mathbf{v}_e = \mathbf{L}^{-1/2}\mathbf{b}_e$ does not change the shape

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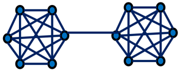

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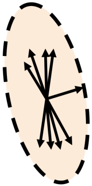

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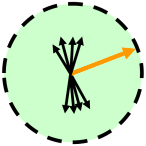

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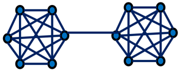

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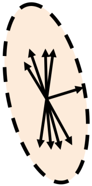

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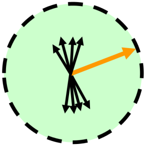

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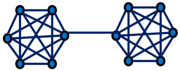
Spectral Graph Sparsification: Intuition

Example: What happens with a dumbbell?

K_n graph



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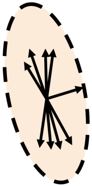

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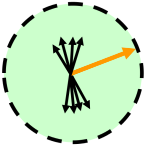

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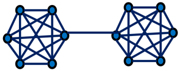

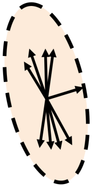

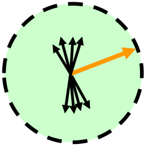

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rescaling reveals the vectors that are critical

https://math.berkeley.edu/~nikhil/

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What it this rescaling $\mathbf{v}_e = \mathbf{L}_G^{-1/2}\mathbf{b}_e$ doing to the norm?

$$\|\mathbf{v}_e\|^2$$

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Edges with higher R_{eff} are more **electrically significant**!

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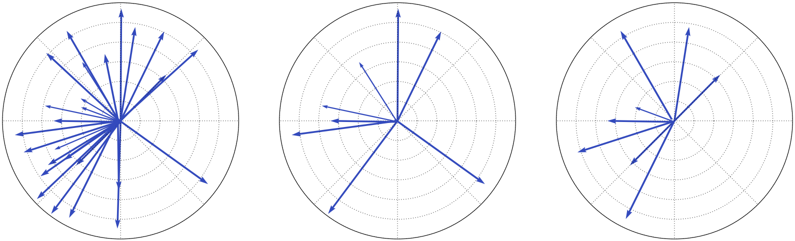
    still unfeasible when m is large

Efficient Sequential Learning
in Structured and Constrained Environments



Without losing information

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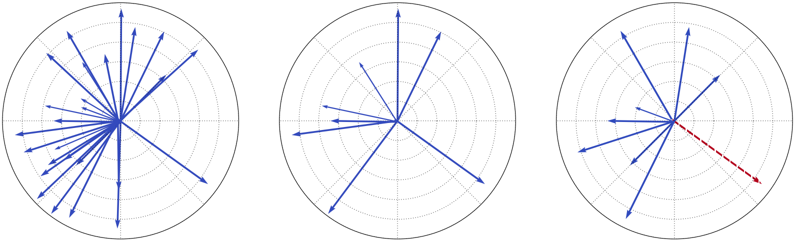

Without losing information

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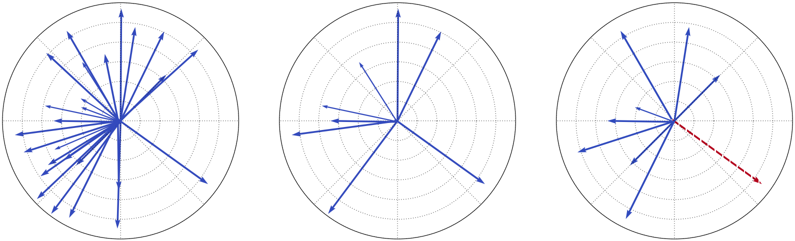

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- data-oblivious methods (e.g., uniform sampling)
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- data-adaptive methods (e.g. eigenvectors, leverage score sampling)
  ↳ accurate but too expensive [Alaoui and Mahoney, 2015]

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Goal 1: find a small, provably accurate dictionary in near-linear time

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new sequential importance sampling approach

↳ analysis for non i.i.d. matrix sampling

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  ↳ adaptive embedding based on KORS dictionary
- preserve fast rates of exact online Newton step
  ↳ new adaptive restart strategy

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Goal 2: use dictionary to solve down-stream problems efficiently

not in this talk: provably accurate solutions in near-linear time

Kernel PCA [Musco and Musco, 2017]

Kernel Regression [Alaoui and Mahoney, 2015; Bach, 2013; Rudi et al., 2015]

Kernel K-Means [Musco and Musco, 2017]

Graph Semi-Supervised Learning [Calandriello et al., 2015]

Graph Sparsification [Calandriello et al., 2016]

Outline

(1) Dictionary learning

- ▷ Nyström sampling
- ▷ ridge leverage scores and effective dimension
- ▷ SQUEAK: sequential RLS importance sampling
  - ↳ analysis for non i.i.d. matrix sampling

(2) Online Kernel Learning

- ▷ online kernel learning and kernelized online Newton step
- ▷ PROS-N-KONS: adaptive Nyström embedding for online kernel learning
- ▷ adaptive restarts
- ▷ regression and classification experiments

Setting

Samples: $\mathbf{x}_i \in \mathcal{X}$ (e.g. $\mathbb{R}^d$)

Feature map: $\varphi(\mathbf{x}_i) : \mathcal{X} \to \mathcal{H} = \phi_i$

Dataset: $\mathcal{D}_n = \{\phi_i\}_{i=1}^n$, $\Phi_n = [\phi_1, \phi_2, \ldots, \phi_n]$

Empirical Kernel Matrix: $\Phi_n^\mathsf{T}\Phi_n = \mathbf{K}_n \in \mathbb{R}^{n \times n}$

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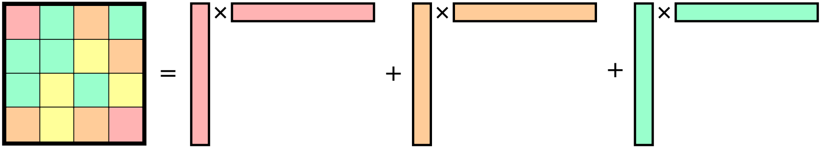
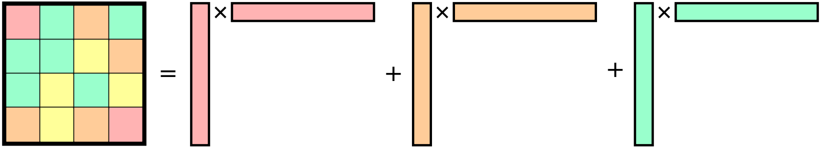


Dictionary Learning

What is Dictionary Learning (DL)?

Representation/Unsupervised learning:



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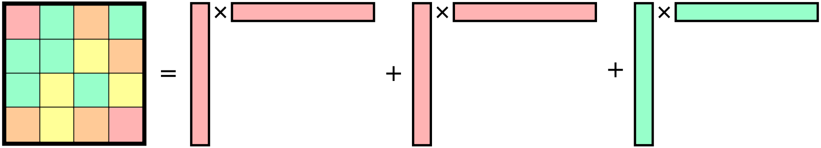

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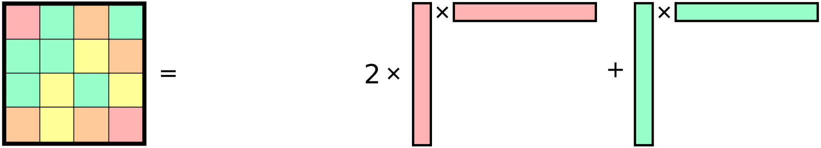

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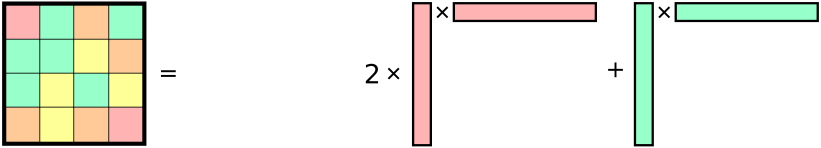

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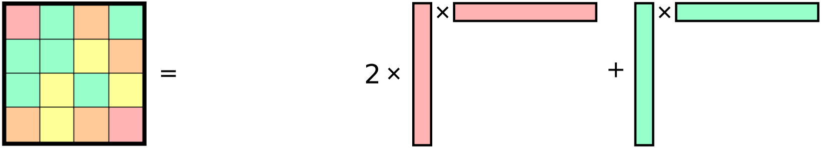

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(1) which to pick?   (2) how many to pick?   (3) how to build $\mathcal{I}$?

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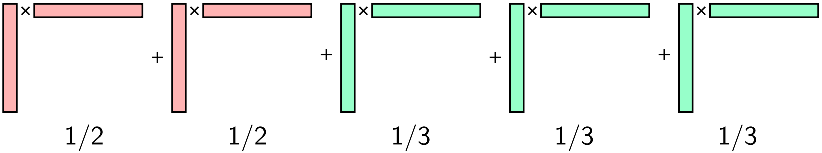

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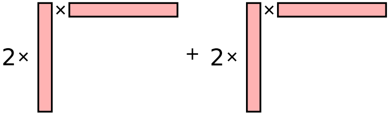

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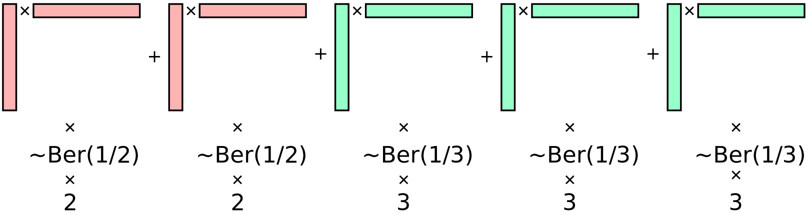

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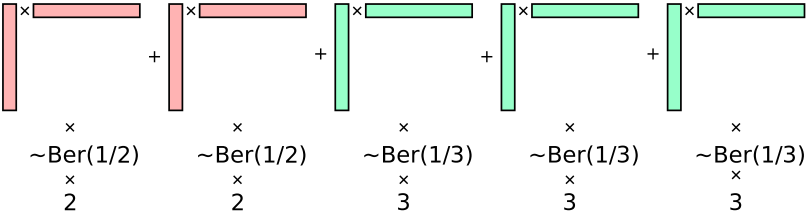

Nyström sampling: unbiased estimator

$$\Phi_n \mathbf{S}_n \mathbf{S}_n^{\mathsf{T}} \Phi_n^{\mathsf{T}} = \sum_{i=1}^{n} \sum_{j=1}^{\overline{q}} \frac{1}{p_i} \frac{z_{i,j}}{\overline{q}} \phi_i \phi_i^{\mathsf{T}}$$

Ridge Leverage Scores

Intuitively, RLS capture orthogonality

$$\tau_{n,i} = \mathbf{e}_{n,i}\mathbf{K}_n^\mathsf{T}(\mathbf{K}_n + \gamma\mathbf{I}_n)^{-1}\mathbf{e}_{n,i} = \phi_i^\mathsf{T}(\Phi_n\Phi_n^\mathsf{T} + \gamma\mathbf{I})^{-1}\phi_i$$

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If all ϕ_i are identical (collinear), we have

$$\tau_{n,i} = \phi_i^\mathsf{T}(n\phi_i\phi_i^\mathsf{T} + \gamma\mathbf{I})^{-1}\phi_i = \frac{\phi_i^\mathsf{T}\phi_i}{n\phi_i^\mathsf{T}\phi_i + \gamma} \sim \frac{1}{n}$$

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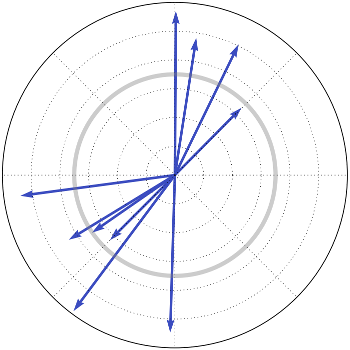
Given Φ_{t-1}, adding a new column to it can only reduce the RLS of columns already in Φ_{t-1}

$$\boldsymbol{\tau_{t,i} \leq \tau_{t-1,i}}$$

Ridge Leverage Scores



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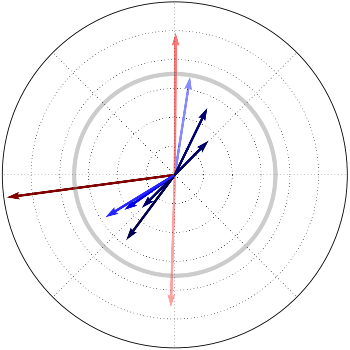
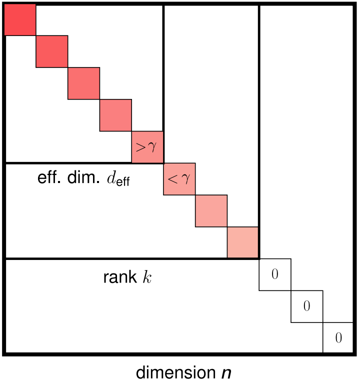


Effective Dimension

Intuitively, the effective dimension is the number of relevant directions in the data



$$d_{\text{eff}}^{n}(\gamma) = \sum_{i=1}^{n} \tau_{n,i} = \operatorname{Tr}\left(\mathbf{K}_n(\mathbf{K}_n + \gamma \mathbf{I}_n)^{-1}\right) = \sum_{i=1}^{n} \frac{\lambda_i(\mathbf{K}_n)}{\lambda_i(\mathbf{K}_n) + \gamma} \leq \operatorname{Rank}(\mathbf{K}_n)$$

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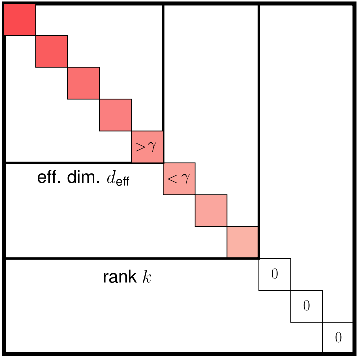

Given $d_{\text{eff}}^{t-1}(\gamma)$, adding a new column to Φ_{t-1} can only increase $d_{\text{eff}}^{t}(\gamma)$

$$\mathbf{d}_{\text{eff}}^{\mathbf{t}}(\gamma) \geq \mathbf{d}_{\text{eff}}^{\mathbf{t-1}}(\gamma)$$

Reconstruction guarantees

An (ε, γ)-accurate dictionary $\mathcal{I}$ satisfies

$$\Phi \mathbf{S}\mathbf{S}^{\mathsf{T}} \Phi^{\mathsf{T}}$$

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Projection $\mathbf{\Pi}_{\mathcal{I}} = \Phi\mathbf{S}(\mathbf{S}^{\mathsf{T}}\Phi^{\mathsf{T}}\Phi\mathbf{S})\mathbf{S}^{\mathsf{T}}\Phi^{\mathsf{T}}$ on dictionary span

↳ Nyström approx. $\widetilde{\mathbf{K}} = \Phi^{\mathsf{T}}\mathbf{\Pi}_{\mathcal{I}}\Phi$

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Graph sparsification (not in this talk)

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Graph sparsification (not in this talk)

In graph problems dictionary $\mathcal{I}$ is subset of reweighted edges

↳ $(1-\varepsilon)\mathbf{L}_{\mathcal{G}} \preceq \mathbf{L}_{\mathcal{I}} \preceq (1+\varepsilon)\mathbf{L}_{\mathcal{G}}$

Oracle RLS Sampling

Theorem (Alaoui and Mahoney, 2015)

Given γ be the Nystrom regularization, ε the accuracy, δ the confidence. If the dictionary $\mathcal{I}_n$ is computed using the sampling distribution $p_{n,i} \propto \tau_{n,i}$ and using at least m columns

$$m \geq \left(\frac{2 d_{eff}^n(\gamma)}{\varepsilon^2} \right) \log \left(\frac{n}{\delta} \right),$$

then with probability $1 - \delta$ we have

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~~Goal 1: small and accurate dictionary~~ done!

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Goal 1: small and accurate dictionary in near-linear time

If someone gave us the RLS

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Computing $\tau_{n,i} = \mathbf{e}_{n,i}\mathbf{K}_n^\mathsf{T}(\mathbf{K}_n + \gamma\mathbf{I}_n)^{-1}\mathbf{e}_{n,i}$ also requires storing and inverting the full $\mathbf{K}_n$

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Given dictionary $\mathcal{I}_n$ with $|\mathcal{I}_n| = J$ atoms

$$\tau_{n,i} = \mathbf{e}_{n,i}\mathbf{K}_t^\mathsf{T}(\mathbf{K}_n + \gamma\mathbf{I}_n)^{-1}\mathbf{e}_{n,i}$$

- $\widetilde{\tau}_{n,i} = \mathbf{e}_i^\mathsf{T}\widetilde{\mathbf{K}}_\mathbf{n}(\widetilde{\mathbf{K}}_\mathbf{t} + \gamma\mathbf{I})^{-1}\mathbf{e}_i$

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- $\widetilde{\tau}_{n,i} = \mathbf{e}_i^\mathsf{T}\widetilde{\mathbf{K}}_\mathbf{n}(\widetilde{\mathbf{K}}_\mathbf{t} + \gamma\mathbf{I})^{-1}\mathbf{e}_i$
- Instead, approximate $\tau_{n,i}$ directly in $\mathcal{H}$

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$$
\begin{aligned}
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\end{aligned}
$$

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- Instead, approximate $\tau_{n,i}$ directly in $\mathcal{H}$, and then use kernel trick
- If $\mathcal{I}$ (ε,γ)-accurate $\Rightarrow \tau_{n,i}(\gamma)/\left(\frac{1+3\varepsilon}{1-\varepsilon}\right) \leq \widetilde{\tau}_{n,i} \leq \tau_{n,i}(\gamma)$
  [Calandriello et al., 2017a]

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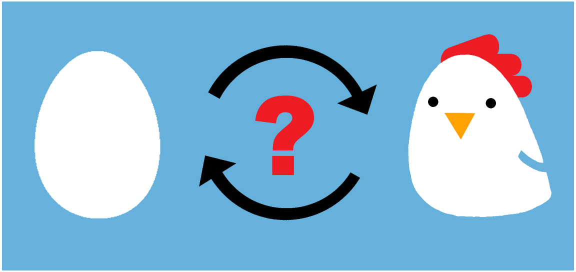
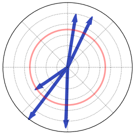
- $\widetilde{\tau}_{n,i}$ for $i \in \mathcal{I}_n$ can be computed using only samples contained in $\mathcal{I}_n$.

Chicken and egg problem



SQUEAK- Sequential RLS sampling



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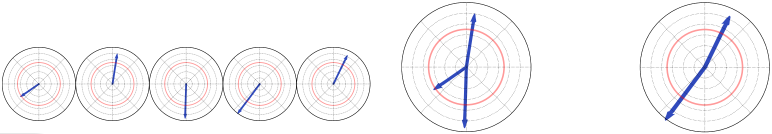

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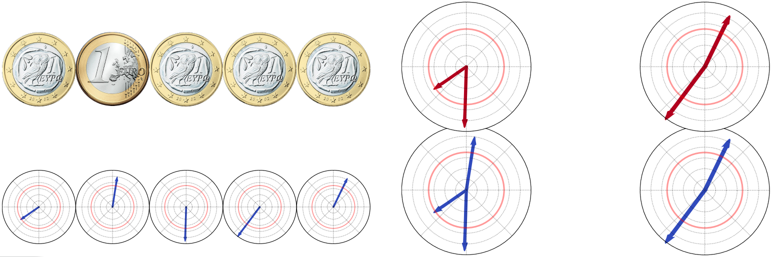
$\widetilde{p}_{1,i} \propto \widetilde{\tau}_{1,i}$,
$z_{1,i} = \mathbb{I}\{Ber(\widetilde{p}_{1,i})\}$



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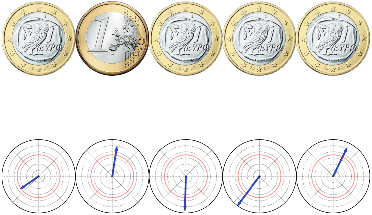
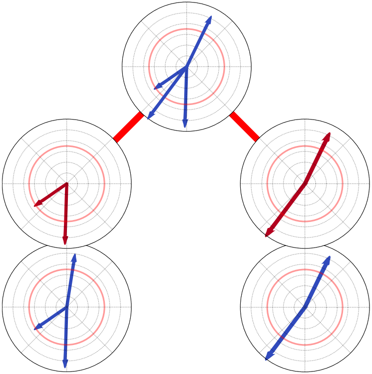




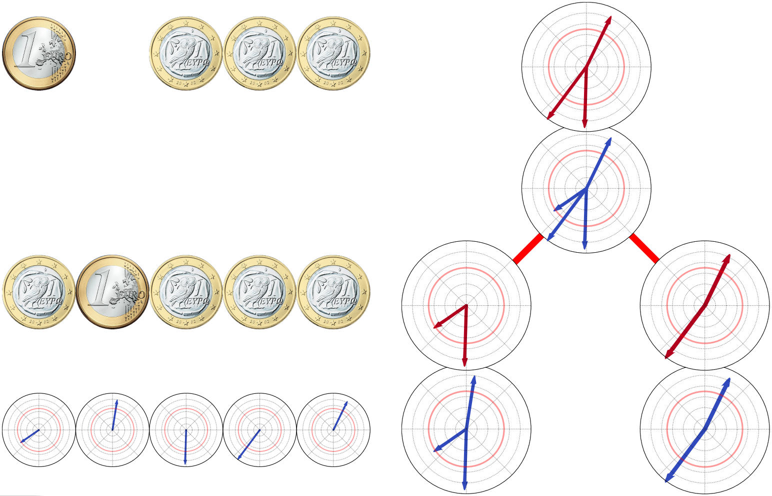
SQUEAK- Sequential RLS sampling

$\widetilde{p}_{1,i} \propto \widetilde{\tau}_{1,i}$,

$z_{1,i} = \mathbb{I}\{Ber(\widetilde{p}_{1,i})\}$

$\widetilde{p}_{2,i} \propto \widetilde{\tau}_{2,i}$

$z_{2,i} = \mathbb{I}\left\{Ber\left(\frac{\widetilde{p}_{2,i}}{\widetilde{p}_{1,i}}\right)\right\} z_{1,i}$



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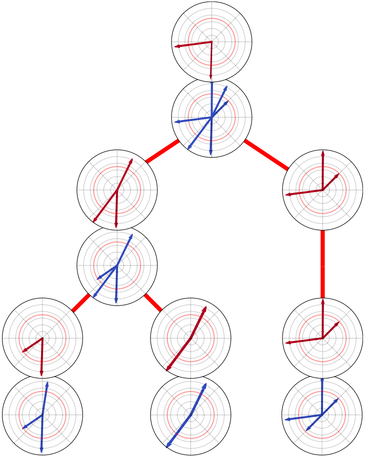
$\widetilde{p}_{3,i} \propto \widetilde{\tau}_{3,i}$ $\quad z_{3,i} = \mathbb{I}\left\{Ber\left(\frac{\widetilde{p}_{3,i}}{\widetilde{p}_{3,i}}\right)\right\} z_{2,i}$



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$\widetilde{p}_{1,i} \propto \widetilde{\tau}_{1,i}$, $z_{1,i} = \mathbb{I}\{Ber(\widetilde{p}_{1,i})\}$

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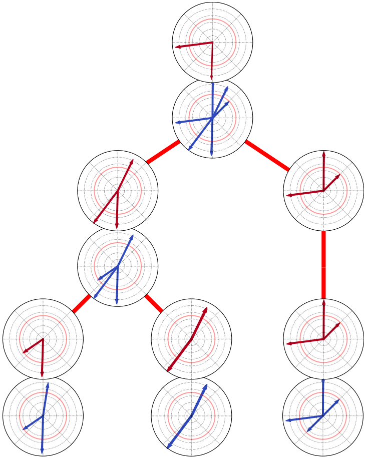
- Store points directly in $\mathcal{I}$
  - ↳ single pass over the dataset



SQUEAK- Sequential RLS sampling

$\widetilde{p}_{1,i} \propto \widetilde{\tau}_{1,i},$
$z_{1,i} = \mathbb{I}\{Ber(\widetilde{p}_{1,i})\}$

$\widetilde{p}_{2,i} \propto \widetilde{\tau}_{2,i}$
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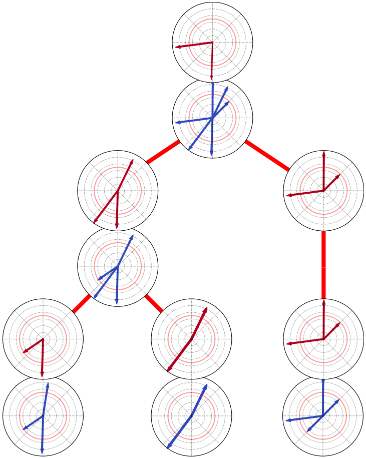

- Store points directly in $\mathcal{I}$
  - ↳ single pass over the dataset
- Unnormalized $\widetilde{p}_{t,i}$
  - ↳ no need for approximate $d_{\text{eff}}(\gamma)_t$

SQUEAK- Sequential RLS sampling

$\widetilde{p}_{1,i} \propto \widetilde{\tau}_{1,i},$
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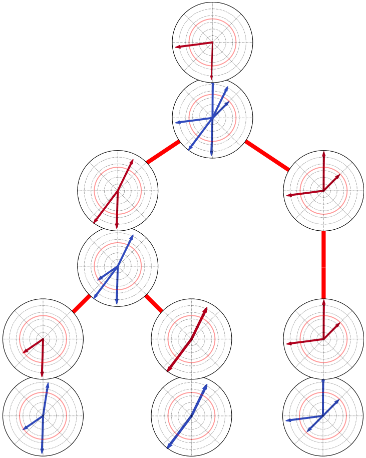

- Store points directly in $\mathcal{I}$
  - ↳ single pass over the dataset
- Unnormalized $\widetilde{p}_{t,i}$
  - ↳ no need for approximate $d_{\text{eff}}(\gamma)_t$
- Never recompute $\widetilde{\tau}_{t,i}$ after dropping
  - ↳ never construct the whole $\mathbf{K}_n$

SQUEAK- Sequential RLS sampling

$\widetilde{p}_{1,i} \propto \widetilde{\tau}_{1,i}$,
$z_{1,i} = \mathbb{I}\{Ber(\widetilde{p}_{1,i})\}$

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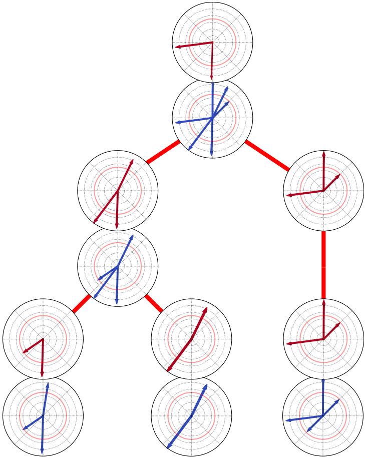

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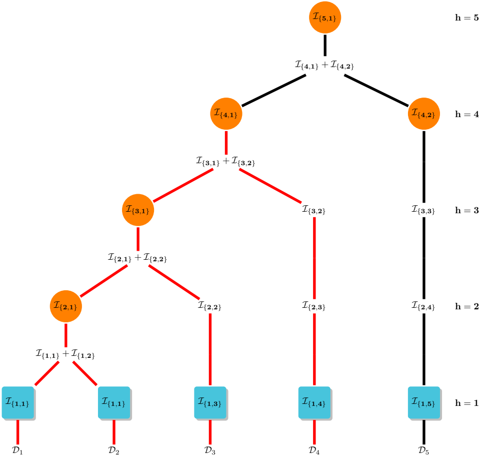
  - ↳ never construct the whole $\mathbf{K}_n$
- Runtime depends on merge tree

SQUEAK- Sequential RLS sampling

$\mathcal{I}$ with $|\mathcal{I}| = J$ atoms, space: $\mathcal{O}(J^2)$, Runtime: single merge $\mathcal{O}(J^3)$

SQUEAK- Sequential RLS sampling

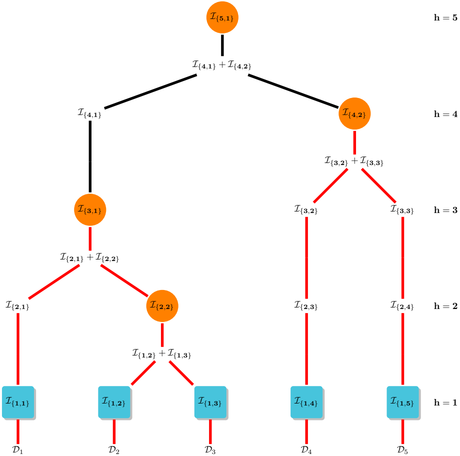
SQUEAK - fully unbalanced tree: $\widetilde{\mathcal{O}}(nJ^3)$



$\mathcal{I}$ with $|\mathcal{I}| = J$ atoms, space: $\mathcal{O}(J^2)$, Runtime: single merge $\mathcal{O}(J^3)$

DISQUEAK- Distributed sequential RLS sampling

DISQUEAK - fully balanced tree: $\widetilde{\mathcal{O}}(\log(n)J^3)$



$\mathcal{I}$ with $|\mathcal{I}| = J$ atoms, space: $\mathcal{O}(J^2)$, Runtime: single merge $\mathcal{O}(J^3)$

DISQUEAK

Theorem (Calandriello et al., 2017a)

Let $\alpha = \left(\frac{1+2\varepsilon}{1-2\varepsilon}\right)$ *and* $\gamma > 1$. *For any* $0 \leq \varepsilon \leq 1$, *and* $0 \leq \delta \leq 1$, *if we run* DISQUEAK *with* $\overline{\mathbf{q}} \geq \frac{\mathbf{26}\alpha}{\varepsilon^2} \log(\frac{\mathbf{n}}{\delta})$, *then w.p.* $1-\delta$, *for all nodes* $\{h, l\}$

(1) *The dictionary* $\mathcal{I}_{\{h,l\}}$ *is* (ε, γ)*-accurate*.

(2) $|\mathcal{I}_{\{\mathbf{h},\mathbf{l}\}}| \leq \mathcal{O}(\overline{q} d_{eff}(\gamma)_{\{h,l\}}) \leq \mathcal{O}(\frac{\alpha}{\varepsilon^2} d_{eff}^n(\gamma) \log(\frac{n}{\delta}))$.

- Accuracy/dictionary size match oracle RLS-sampling at any time
  ↳ no free lunch: space/time scale with $|\mathcal{I}| \leq d_{\text{eff}}^n(\gamma)$

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- Merge tree fixed in advance

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- Runtime: single merge $\mathcal{O}(|\mathcal{I}_n|^3) \leq \widetilde{\mathcal{O}}(d^n_{\text{eff}}(\gamma)^3)$
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- Fully balanced tree: $\widetilde{\mathcal{O}}(\log(n) d^n_{\text{eff}}(\gamma)^3)$ time, $\widetilde{\mathcal{O}}(n d^n_{\text{eff}}(\gamma)^3)$ work!

Comparison

 = oracle, $\mu(\gamma) = \max_i \tau_{n,i}(\gamma) \leq 1/\gamma$ regularized coherence

	$\widetilde{\mathcal{O}}$(Runtime)	$\mathcal{O}(\lvert\mathcal{I}_n\rvert)$	Passes
Bach, 2013 (Uniform)	$n\mu(\gamma) +$ 	$n\mu(\gamma)$	1

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Alaoui and Mahoney, 2015	$n^3\mu(\gamma)^2$	$n\mu(\gamma) + d_{\text{eff}}^n(\gamma)\log(n)$	3

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SQUEAK/DISQUEAK Calandriello et al., 2017a	$(n/k)d_{\text{eff}}^n(\gamma)^3$	$d_{\text{eff}}^n(\gamma)\log(n)$	1

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SQUEAK/DISQUEAK Calandriello et al., 2017a	$(n/k)d_{\text{eff}}^n(\gamma)^3$	$d_{\text{eff}}^n(\gamma)\log(n)$	1
KORS Calandriello et al., 2017c	$nd_{\text{eff}}^n(\gamma)^2$	$d_{\text{eff}}^n(\gamma)\log^2(n)$	1

Comparison

 = oracle, $\mu(\gamma) = \max_i \tau_{n,i}(\gamma) \leq 1/\gamma$ regularized coherence

	$\widetilde{\mathcal{O}}$(Runtime)	$\mathcal{O}(\lvert\mathcal{I}_n\rvert)$	Passes
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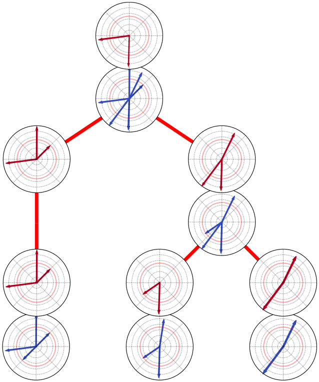
SQUEAK/DISQUEAK Calandriello et al., 2017a	$(n/k)d_{\text{eff}}^n(\gamma)^3$	$d_{\text{eff}}^n(\gamma)\log(n)$	1
KORS Calandriello et al., 2017c	$nd_{\text{eff}}^n(\gamma)^2$	$d_{\text{eff}}^n(\gamma)\log^2(n)$	1
Musco and Musco, 2017	$nd_{\text{eff}}^n(\gamma)^2$	$d_{\text{eff}}^n(\gamma)\log(n)$	$\log(n)$

Proof sketch



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dependent chains

of dependent coin flip

Proof sketch

Similar to importance sampling. If the $\widetilde{p}_{t,i}$ were fixed in advance

$$\mathbb{P}(z_{t,i,j} = 1) = \mathbb{P}(\mathcal{B}(\widetilde{p}_{t,i}/\widetilde{p}_{t-1,i}) = 1)\mathbb{P}(z_{t-1,i,j} = 1)$$

Proof sketch

Need to bound

$$\mathbb{P}\left(\exists t \in \{1, \ldots, n\} : \|\mathbf{P}_t - \widetilde{\mathbf{P}}_t\|_2 \geq \varepsilon \cup |\mathcal{I}_t| \geq 3\overline{q} d_{\text{eff}}(\gamma)_t\right)$$

Proof sketch

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$$\mathbb{P}\left(\exists t \in \{1,\ldots,n\} : \|\mathbf{P}_t - \widetilde{\mathbf{P}}_t\|_2 \geq \varepsilon \cup |\mathcal{I}_t| \geq 3\overline{q}d_{\text{eff}}(\gamma)_t\right)$$

After a union bound

$$\sum_{t=1}^{n} \mathbb{P}\left(\|\mathbf{P}_t - \widetilde{\mathbf{P}}_t\|_2 \geq \varepsilon\right)$$
$$+ \sum_{t=1}^{n} \mathbb{P}\left(|\mathcal{I}_t| \geq 3\overline{q}d_{\text{eff}}(\gamma)_t \cap \left\{\forall t' \in \{1,\ldots,t\} : \|\mathbf{P}_t - \widetilde{\mathbf{P}}_t\|_2 \leq \varepsilon\right\}\right)$$

Proof sketch

We start by bounding $\mathbb{P}\left(\|\mathbf{P}_t - \widetilde{\mathbf{P}}_t\|_2 \geq \varepsilon\right)$. Let

$$z_{s,i,j} = \mathbb{I}\left\{u_{s,i,j} \leq \frac{\widetilde{p}_{s,i}}{\widetilde{p}_{s-1,i}}\right\} z_{s-1,i,j}, \qquad \mathbf{v}_i = (\mathbf{K}_t + \gamma \mathbf{I})^{-1} \mathbf{K}_t^{1/2} \mathbf{e}_{t,i}$$

with $u_{s,i,j} \sim \mathcal{U}(0,1)$. Then

$$\mathbf{Y}_t = \mathbf{P}_t - \widetilde{\mathbf{P}}_t = \frac{1}{\overline{q}} \sum_{i=1}^{t} \sum_{j=1}^{\overline{q}} \left(1 - \frac{z_{t,i,j}}{\widetilde{p}_{t,i}}\right) \mathbf{v}_i \mathbf{v}_i^{\mathsf{T}}$$

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Cannot use concentrations for independent r.v., because $z_{t,i,j}$ and $z_{t,i',j'}$ are both dependent on $z_{t-1,i'',j''}$ through the estimates.

Proof sketch

Build the martingale

$$\mathbf{X}_{\{s,i,j\}} = \left(\frac{z_{s-1,i,j}}{\widetilde{p}_{s-1,i}} - \frac{z_{t,i,j}}{\widetilde{p}_{s,i}} \right) \mathbf{v}_i \mathbf{v}_i^{\mathsf{T}}$$

We can use variants of Bernstein's inequality for matrix martingales, we need a bound on the range

$$\begin{aligned}
\|\mathbf{X}_{\{s,i,j\}}\| &= \frac{1}{q} \left| \left(\frac{z_{s-1,i,j}}{\widetilde{p}_{s-1,i}} - \frac{z_{s,i,j}}{\widetilde{p}_{s,i}} \right) \right| \|\mathbf{v}_i \mathbf{v}_i^{\mathsf{T}}\| \leq \frac{1}{q} \frac{1}{\widetilde{p}_{s,i}} \|\mathbf{v}_i\|^2 \\
&\leq \frac{1}{q} \frac{1}{\widetilde{p}_{s,i}} \mathbf{v_i}^{\mathsf{T}} \mathbf{v_i} = \frac{1}{q} \frac{1}{\widetilde{p}_{s,i}} \mathbf{e_i}^{\mathsf{T}} \mathbf{K_t}^{1/2} (\mathbf{K_t} + \gamma \mathbf{I})^{-1} \mathbf{K_t}^{1/2} \mathbf{e_i} \\
&= \frac{1}{q} \frac{1}{\widetilde{p}_{s,i}} \mathbf{e_i}^{\mathsf{T}} \mathbf{P_t} \mathbf{e_i} = \frac{1}{q} \frac{\tau_{\mathbf{t,i}}}{\widetilde{p}_{s,i}} \leq \frac{\alpha}{q} \frac{\tau_{t,i}}{p_{s,i}} = \frac{\alpha}{q} \frac{\tau_{t,i}}{\tau_{s,i}} \leq \frac{\alpha}{q} := R,
\end{aligned}$$

Proof sketch

Build the martingale

$$\mathbf{X}_{\{s,i,j\}} = \left(\frac{z_{s-1,i,j}}{\widetilde{p}_{s-1,i}} - \frac{z_{t,i,j}}{\widetilde{p}_{s,i}} \right) \mathbf{v}_i \mathbf{v}_i^\mathsf{T}$$

We can use variants of Bernstein's inequality for matrix martingales, we need a bound on the range

$$\begin{aligned}
\|\mathbf{X}_{\{s,i,j\}}\| &= \frac{1}{q} \left| \left(\frac{z_{s-1,i,j}}{\widetilde{p}_{s-1,i}} - \frac{z_{s,i,j}}{\widetilde{p}_{s,i}} \right) \right| \|\mathbf{v}_i \mathbf{v}_i^\mathsf{T}\| \leq \frac{1}{q} \frac{1}{\widetilde{p}_{s,i}} \|\mathbf{v}_i\|^2 \\
&\leq \frac{1}{q} \frac{1}{\widetilde{p}_{s,i}} \mathbf{v_i^\mathsf{T} v_i} = \frac{1}{q} \frac{1}{\widetilde{p}_{s,i}} \mathbf{e_i^\mathsf{T} K_t^{1/2} (K_t + \gamma I)^{-1} K_t^{1/2} e_i} \\
&= \frac{1}{q} \frac{1}{\widetilde{p}_{s,i}} \mathbf{e_i^\mathsf{T} P_t e_i} = \frac{1}{q} \frac{\tau_{\mathbf{t,i}}}{\widetilde{p}_{s,i}} \leq \frac{\alpha}{q} \frac{\tau_{t,i}}{p_{s,i}} = \frac{\alpha}{q} \frac{\tau_{t,i}}{\tau_{s,i}} \leq \frac{\alpha}{q} := R,
\end{aligned}$$

RLS normalize our r.v.

Proof sketch

Now bound the total variation

$$\mathbf{W} = \sum \mathbb{E}\left[\mathbf{X}_{\{s,i,j\}}^2 \;\middle|\; \{\mathbf{X}_r\}_{r=0}^{\{s,i,j\}-1}\right]$$

$$= \frac{1}{\overline{q}^2} \sum_{j=1}^{\overline{q}} \sum_{i=1}^{t} \sum_{s=1}^{t} \frac{z_{s-1,i,j}}{\widetilde{p}_{s-1,i}} \left(\frac{1}{\widetilde{p}_{s,i}} - \frac{1}{\widetilde{p}_{s-1,i}}\right) \mathbf{v}_i \mathbf{v}_i^\mathsf{T} \mathbf{v}_i \mathbf{v}_i^\mathsf{T}$$

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Deterministically

$$\|\mathbf{W}\| = \left\| \frac{1}{\overline{q}^2} \sum_{j=1}^{\overline{q}} \sum_{i=1}^{t} \sum_{s=1}^{t} \frac{z_{s-1,i,j}}{\widetilde{p}_{s-1,i}} \left(\frac{1}{\widetilde{p}_{s,i}} - \frac{1}{\widetilde{p}_{s-1,i}}\right) \mathbf{v}_i \mathbf{v}_i^\mathsf{T} \mathbf{v}_i \mathbf{v}_i^\mathsf{T} \right\|$$
$$\leq \left\| \frac{1}{\overline{q}^2} \sum_{j=1}^{\overline{q}} \sum_{i=1}^{t} \frac{\mathbf{v}_i^\mathsf{T} \mathbf{v}_i}{\widetilde{p}_{t,i}^2} \mathbf{v}_i \mathbf{v}_i^\mathsf{T} \right\| \leq \left\| \frac{\alpha}{\overline{q}} \sum_{i=1}^{t} \frac{1}{\widetilde{p}_{t,i}} \mathbf{v}_i \mathbf{v}_i^\mathsf{T} \right\|$$
$$\leq \left\| \frac{\alpha^2}{\overline{q}} \sum_{i=1}^{t} \mathbf{I} \right\| = \frac{\alpha^2}{\overline{q}} t$$

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Deterministic bound on variance too large

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$$\mathbb{P}\left(\|\mathbf{Y}_t\| \geq \varepsilon \ \cap \ \|\mathbf{W}\| \leq \sigma^2\right) \leq t \exp\{-\dots\}$$

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Starting from an upper bound on $\mathbf{W}$ that is still a r.v.

$$\mathbf{W} \preceq \frac{1}{\overline{q}^2} \sum_{j=1}^{\overline{q}} \sum_{i=1}^{t} \max_{s=0}^{t-1} \left\{ \frac{z_{s,i,j}}{\widetilde{p}_{s,i}^2} \right\} \mathbf{v}_i \mathbf{v}_i^{\mathsf{T}} \mathbf{v}_i \mathbf{v}_i^{\mathsf{T}}$$

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This still has high variance: cannot simply apply martingale Bernstein

Proof sketch

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We will find another set of dominating r.v. $1/w_{i,j}$, indep. from each other
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Random variable A stochastically dominates random variable B, if for all values a the two equivalent conditions are verified

$$\mathbb{P}(A \geq a) \geq \mathbb{P}(B \geq a) \Leftrightarrow \mathbb{P}(A \leq a) \leq \mathbb{P}(B \leq a).$$

Proof sketch

Similar to importance sampling. If the $\widetilde{p}_{t,i}$ were fixed in advance

$$\begin{aligned}\mathbb{P}(z_{t,i,j} = 1) &= \mathbb{P}(\mathcal{B}(\widetilde{p}_{t,i}/\widetilde{p}_{t-1,i}) = 1)\mathbb{P}(z_{t-1,i,j} = 1) \\ &= \frac{\widetilde{p}_{t,i}}{\widetilde{p}_{t-1,i}}\mathbb{P}(z_{t-1,i,j} = 1)\end{aligned}$$

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$$

Weight increase along chain $\frac{z_{t-1,i,j}}{\widetilde{p}_{t-1,i}} \leq \frac{z_{t,i,j}}{\widetilde{p}_{t,i}}$ until $z_{t,i,j} = 0$ or $\frac{1}{\widetilde{p}_{n,i}} \lessapprox \frac{1}{\tau_{n,i}}$.

Proof sketch

Predictable quadratic variation $\mathbf{W}$ of a chain scales (roughly) with

$$\|\mathbf{W}\|_2^2 \sim \max_{s=0}^{t-1} \left\{ \frac{z_{s,i,j}}{\widetilde{p}_{s,i}} \right\}$$

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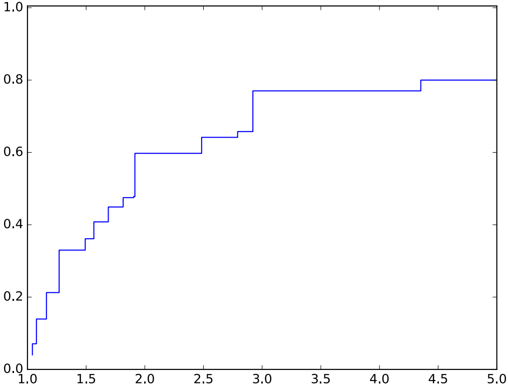

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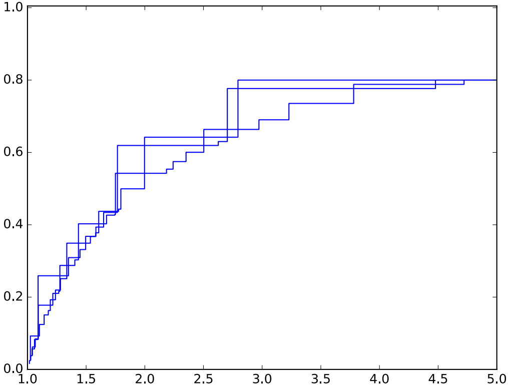

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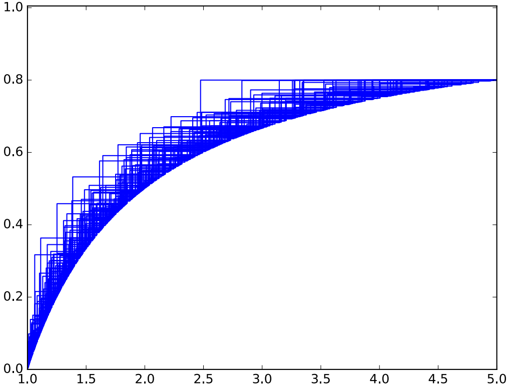

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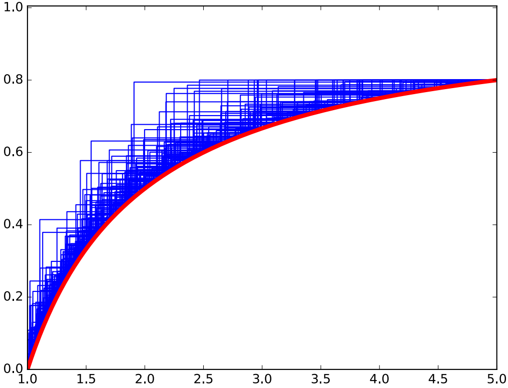

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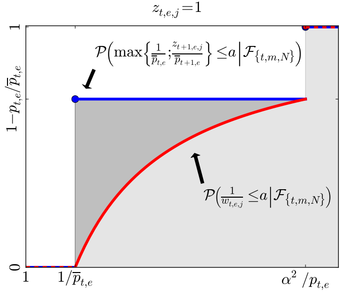

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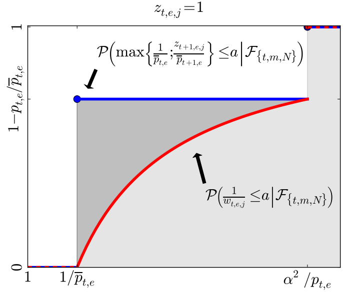

$$\mathbb{P}\left(\max\left\{\frac{z_{s,i,j}}{\widetilde{p}_{s,i}}\right\} \leq a\right) \geq \mathbb{P}\left(\frac{1}{w_{0,i,j}} \leq a\right) = \begin{cases} 0 & \text{for} \quad a < 1 \\ 1 - \frac{1}{a} & \text{for} \quad 1 \leq a < \alpha/p_{t,i} \\ 1 & \text{for} \quad \alpha/p_{t,i} \leq a \end{cases},$$

SQUEAK- recap before application

Goal 1: find a small, provably accurate dictionary in near-linear time

SQUEAK and DISQUEAK

Sub-linear time using multiple machines

Final dictionary can be updated if new samples arrive

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Future work

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  - ↳ Easy to implement: distributed task queue
    Preliminary results promising, easily scales to 1M+ samples

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- Beyond passive processing: SQUEAK for active learning

Efficient Sequential Learning

in Structured and Constrained Environments

Goal 2: use dictionary to solve down-stream problems efficiently

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Low-rank PSD matrix approximation

Kernel matrix $\mathbf{K}_n$

Kernel PCA

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Graph Sparsification

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Batch Conjugate gradient

[Rudi et al., 2017]

Online Newton Step (second part of talk)

[Calandriello et al., 2017b; Calandriello et al., 2017c]

Outline

(1) Dictionary learning

- ▷ Nyström sampling
- ▷ ridge leverage scores and effective dimension
- ▷ SQUEAK: sequential RLS importance sampling
  - ↳ analysis for non i.i.d. matrix sampling

(2) Online Kernel Learning

- ▷ online kernel learning and kernelized online Newton step
- ▷ PROS-N-KONS: adaptive Nyström embedding for online kernel learning
- ▷ adaptive restarts
- ▷ regression and classification experiments

Online Kernel Learning (OKL)

Online game between learner and adversary, at each round $t \in [T]$

1. the adversary reveals a new point $\varphi(\mathbf{x}_t) = \phi_t \in \mathcal{H}$
2. the learner chooses a function $f_{\mathbf{w}_t}$ and predicts $f_{\mathbf{w}_t}(\mathbf{x}_t) = \varphi(\mathbf{x}_t)^\mathsf{T}\mathbf{w}_t$,
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t parameters $\Rightarrow$ $\mathcal{O}(t)$ per-step prediction cost

$\mathbf{g}_t = \ell_t'(\phi_t^\mathsf{T}\mathbf{w}_t)\phi_t := \dot{g}_t\phi_t$

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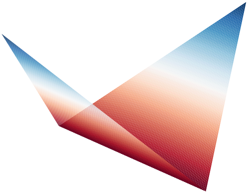
Learning to minimize regret $R(\mathbf{w}) = \sum_{t=1}^T \ell_t(\phi_t\mathbf{w}_t) - \ell_t(\phi_t\mathbf{w})$
and compete with best-in-hindsight $\mathbf{w}^* := \arg\min_{\mathbf{w}\in\mathcal{H}} \sum_{t=1}^T \ell_t(\phi_t\mathbf{w})$

OGD and losses



convex

First order (GD) [Kivinen et al., 2004; Zinkevich, 2003]

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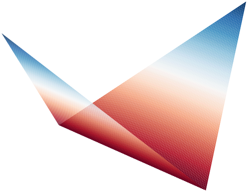

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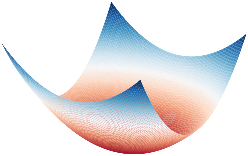

strongly convex

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$\sqrt{T}$ regret, $\mathcal{O}(d)/\mathcal{O}(t)$ time/space per-step

First order (GD) [Hazan et al., 2008]
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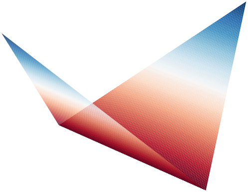
$\sqrt{T}$ regret, $\mathcal{O}(d)/\mathcal{O}(t)$ time/space per-step

First order (GD) [Hazan et al., 2008]
$\log(T)$ regret, but often not satisfied in practice
↳(e.g. $(y_t - \phi_t^{\mathsf{T}} \mathbf{w}_t)^2$)

OGD and losses



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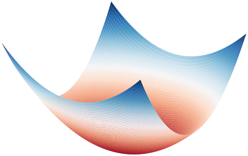

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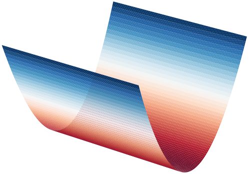
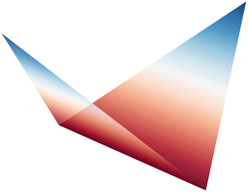


σ**-curved**

Second order (Newton-like) [Hazan et al., 2006; Zhdanov and Kalnishkan, 2010]

$\log(T)$ regret, $\mathcal{O}(d^2)/\mathcal{O}(t^2)$ time/space per-step

OGD and losses



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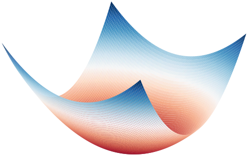

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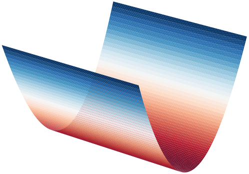

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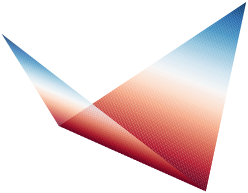

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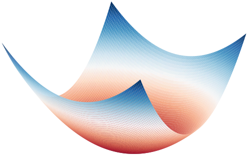

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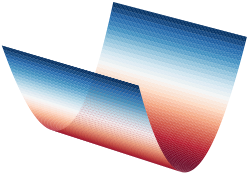

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Assumptions:
ℓ_t are σ-curved and $|\ell_t'(z)| \leq L$ whenever $|z| \leq C$ (scalar Lipschitz)

Second-Order OKL (Kernel Online Newton Step)

Second-Order Gradient Descent

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \mathbf{A}_t^{-1}\mathbf{g}_t, \qquad \mathbf{A}_t = \sum_{s=1}^{t} \sigma \mathbf{g}_s \mathbf{g}_s^\mathsf{T} + \alpha \mathbf{I} = \mathbf{G}_t \mathbf{G}_t^\mathsf{T} + \alpha \mathbf{I}$$

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Regret [Hazan et al., 2006; Luo et al., 2016]

$$R(\mathbf{w}^*) \leq \overbrace{\alpha \|\mathbf{w}^* - \mathbf{w_0}\|_2^2}^{\text{initial error}} + \mathcal{O}\left(\sum_{t=1}^{T} \mathbf{g}_t^\mathsf{T} (\mathbf{G}_t \mathbf{G}_t^\mathsf{T} + \alpha \mathbf{I})^{-1} \mathbf{g}_t \right)$$

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Effective Dimension in online learning

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$d_{\text{eff}}^T(\alpha)$ number of relevant orthogonal directions played by the adversary.

Every new orthogonal direction causes some regret.
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$$d_{\text{eff}}^T(1) \sim \mathcal{O}(1) \leq r$$

is constant in T and

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Approximating KONS

KONS: $d_{\text{eff}}^T(\alpha)\log(T)$ regret

↳ large $\mathcal{H} \Rightarrow \mathcal{O}(t)$ prediction $\phi_t^\mathsf{T}\mathbf{w}_t$, $\mathcal{O}(t^2)$ updates $\mathbf{g}_t - \mathbf{A}_t^{-1}\mathbf{g}_t$

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(a) Exact KONS in $\widetilde{\mathcal{H}}$: $d_{\text{eff}}^T(\alpha)\log(T)$

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(b) error between $\overline{\mathbf{w}}$ best in $\widetilde{\mathcal{H}}$ and $\mathbf{w}^*$ best in $\mathcal{H}$: bound how?

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$\widetilde{\mathcal{H}}$ cannot be fixed

↳ the adversary will find orthogonal points and exploit this

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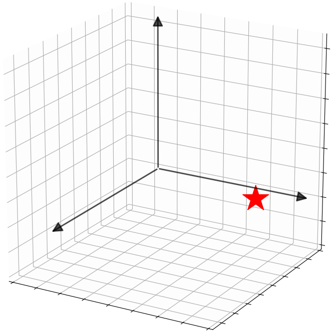
$\widetilde{\mathcal{O}}(d_{\mathrm{eff}}^T(\gamma)^2)$ time/space cost to run exact KONS in $\widetilde{\mathcal{H}}_t$

PROS-N-KONS



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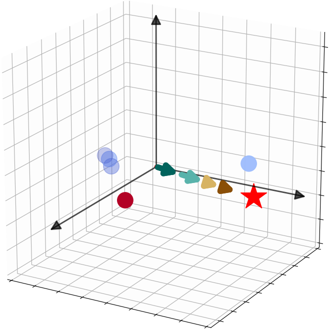

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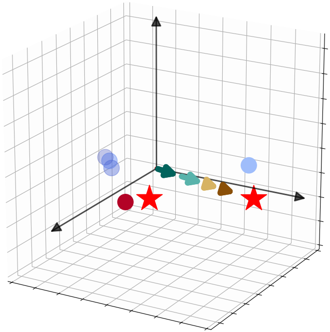

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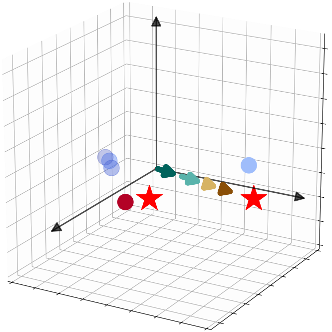
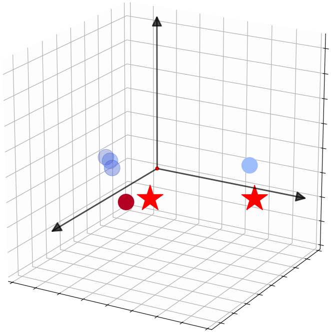


Every time we change $\widetilde{\mathcal{H}}$ we pay $\alpha\|\overline{\mathbf{w}}_j - \mathbf{w}_{t_j}\|_2^2$ (initial error in GD)

↳ the adversary can influence $\mathbf{w}_{t_j}$ and make it large

PROS-N-KONS



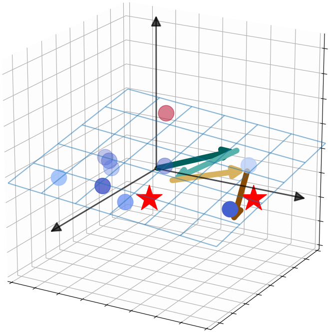
Reset $\widetilde{\mathbf{w}}_t$ and $\widetilde{\mathbf{A}}_t$ when $\widetilde{\mathcal{H}}_t$ changes

↳ wasteful, but not too often. At most $J \leq d_{\text{eff}}^T(\gamma)$ times.

learning is preserved through $\widetilde{\mathcal{H}}_t$ that always improves

adaptive doubling trick

PROS-N-KONS



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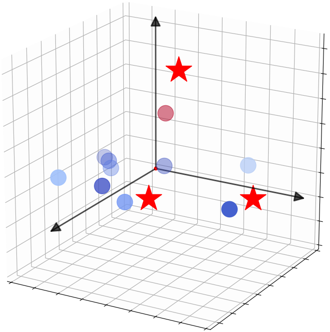

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Final regret guarantees

For any curved loss

$$R_T(\mathbf{w}) \leq \mathcal{O}\Big(\underbrace{d_{\text{eff}}^T(\gamma)\log^2(T)}_{\text{restarts}}(\alpha\|\mathbf{w}\|^2 + \underbrace{d_{\text{eff}}^T(\alpha)\log(T/\alpha)}_{\text{online-offline gap}}) + \underbrace{\gamma T}_{\mathcal{H}\text{-}\widetilde{\mathcal{H}}\text{ gap}} /\alpha\Big),$$

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Setting $\gamma = \alpha/T$ removes second term

↳ regret/computational cost is $\widetilde{\mathcal{O}}(d_{\text{eff}}^T(1/T)^2)$

Final regret guarantees

For any curved loss

$$R_T(\mathbf{w}) \leq \mathcal{O}\Big(\underbrace{d_{\text{eff}}^T(\gamma)\log^2(T)}_{\text{restarts}}(\alpha\|\mathbf{w}\|^2 + \underbrace{d_{\text{eff}}^T(\alpha)\log(T/\alpha)}_{\text{online-offline gap}}) + \underbrace{\gamma T}_{\mathcal{H}\text{-}\widetilde{\mathcal{H}}\text{ gap}}/\alpha\Big),$$

Setting $\gamma = \alpha/T$ removes second term

↳ regret/computational cost is $\widetilde{\mathcal{O}}(d_{\text{eff}}^T(1/T)^2)$

still small in many cases, scale with eigenvalue decay

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- If $\lambda_t = e^{-t}$ (Gaussian $\mathcal{H}$), regret is $o(\text{polylog}(T))$

Final regret guarantees

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- If $\lambda_t = e^{-t}$ (Gaussian $\mathcal{H}$), regret is $o(\text{polylog}(T))$
- If $\mathcal{H} = \mathbb{R}^d$ regret is $\mathcal{O}(r\log(T))$ [Luo et al., 2016]

Final regret guarantees

For squared loss only and $\gamma = \alpha$

$$R(\mathbf{w}^*) \leq \widetilde{\mathcal{O}}\left(J\left(\alpha\|\mathbf{w}^*\|_2^2 + d_{\text{eff}}^T(\alpha)\log(T/\alpha)\right) + J\mathcal{L}^*\right)$$

Final regret guarantees

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Last term $\mathcal{L}^* = \sum_{t=1}^{T} \ell_t(\phi_t \mathbf{w}^*) + \alpha\|\mathbf{w}^*\|_2^2$ replaces $\frac{\gamma}{\alpha}T$

↳ regularized cumulative loss of $\mathbf{w}^*$, very small if $\mathcal{H}$ is good

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First-order regret bound, $\mathcal{L}^*$ constant if model is correct

↳ constant $\mathcal{H}$-$\widetilde{\mathcal{H}}$ gap is enough if instantaneous loss goes to 0.

Experiments - regression

$\alpha = 1, \gamma = 1$						
Algorithm	cadata $n = 20k$, $d = 8$			casp $n = 45k$, $d = 9$		
	Avg. Squared Loss	#SV	Time	Avg. Squared Loss	#SV	Time
FOGD	0.04097 $\pm$ 0.00015	30	—	0.08021 $\pm$ 0.00031	30	—
NOGD	0.03983 $\pm$ 0.00018	30	—	0.07844 $\pm$ 0.00008	30	—
PROS-N-KONS	0.03095 $\pm$ 0.00110	20	18.59	**0.06773** $\pm$ 0.00105	21	40.73
CON-KONS	**0.02850** $\pm$ 0.00174	19	18.45	**0.06832** $\pm$ 0.00315	20	40.91
B-KONS	0.03095 $\pm$ 0.00118	19	18.65	**0.06775** $\pm$ 0.00067	21	41.13
BATCH	0.02202 $\pm$ 0.00002	—	—	0.06100 $\pm$ 0.00003	—	—

Algorithm	slice $n = 53k$, $d = 385$			year $n = 463k$, $d = 90$		
	Avg. Squared Loss	#SV	Time	Avg. Squared Loss	#SV	Time
FOGD	**0.00726** $\pm$ 0.00019	30	—	0.01427 $\pm$ 0.00004	30	—
NOGD	0.02636 $\pm$ 0.00460	30	—	0.01427 $\pm$ 0.00004	30	—
DUAL-SGD	—	—	—	0.01440 $\pm$ 0.00000	100	—
PROS-N-KONS	did not complete	—	—	0.01450 $\pm$ 0.00014	149	884.82
CON-KONS	did not complete	—	—	0.01444 $\pm$ 0.00017	147	889.42
B-KONS	**0.00913** $\pm$ 0.00045	100	60	**0.01302** $\pm$ 0.00006	100	505.36
BATCH	0.00212 $\pm$ 0.00001	—	—	0.01147 $\pm$ 0.00001	—	—

Experiments - binary classification

$\alpha = 1,\ \gamma = 1$						
Algorithm	ijcnn1 $n = 141,691,\ d = 22$			cod-rna $n = 271,617,\ d = 8$		
	accuracy	#SV	time	accuracy	#SV	time
FOGD	9.06 ± 0.05	400	—	10.30 ± 0.10	400	—
NOGD	9.55 ± 0.01	100	—	13.80 ± 2.10	100	—
DUAL-SGD	**8.35** ± 0.20	100	—	**4.83** ± 0.21	100	—
PROS-N-KONS	9.70 ± 0.01	100	211.91	13.95 ± 1.19	38	270.81
CON-KONS	9.64 ± 0.01	101	215.71	18.99 ± 9.47	38	271.85
B-KONS	9.70 ± 0.01	98	206.53	13.99 ± 1.16	38	274.94
BATCH	8.33 ± 0.03	—	—	3.781 ± 0.01	—	—

$\alpha = 0.01,\ \gamma = 0.01$						
Algorithm	ijcnn1 $n = 141,691,\ d = 22$			cod-rna $n = 271,617,\ d = 8$		
	accuracy	#SV	time	accuracy	#SV	time
FOGD	9.06 ± 0.05	400	—	10.30 ± 0.10	400	—
NOGD	9.55 ± 0.01	100	—	13.80 ± 2.10	100	—
DUAL-SGD	8.35 ± 0.20	100	—	4.83 ± 0.21	100	—
PROS-N-KONS	10.73 ± 0.12	436	1003.82	4.91 ± 0.04	111	459.28
CON-KONS	6.23 ± 0.18	432	987.33	5.81 ± 1.96	111	458.90
B-KONS	**4.85** ± 0.08	100	147.22	**4.57** ± 0.05	100	333.57
BATCH	5.61 ± 0.01	—	—	3.61 ± 0.01	—	—

PROS-N-KONS - recap

Goal 2: use dictionary to solve down-stream problems efficiently

PROS-N-KONS: avoid curse of kernelization, constant per-step cost

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Future work

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Future work

Restarts really necessary?

Adaptive α and γ?

Conclusions

Goal 1: find a small, provably accurate dictionary in near-linear time

SQUEAK and DISQUEAK
↳match space/accuracy of oracle RLS sampling
linear or sublinear runtime, single-pass

Goal 2: use dictionary to solve down-stream problems efficiently

PROS-N-KONS
↳preserve logarithmic rate with constant per-step cost

Leverage existing analysis to get provably accurate linear-time algorithms

Open questions

Short-term: more applications, more experiments

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Kernel Ridge Regression - Gaussian Process - Laplacian Smoothing

Kernel PCA - Graph Spectral Embedding

Empirically: which kernel/γ for which dataset/α

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Long-term: new problems

Deterministic algorithms [Ghashami et al., 2015]

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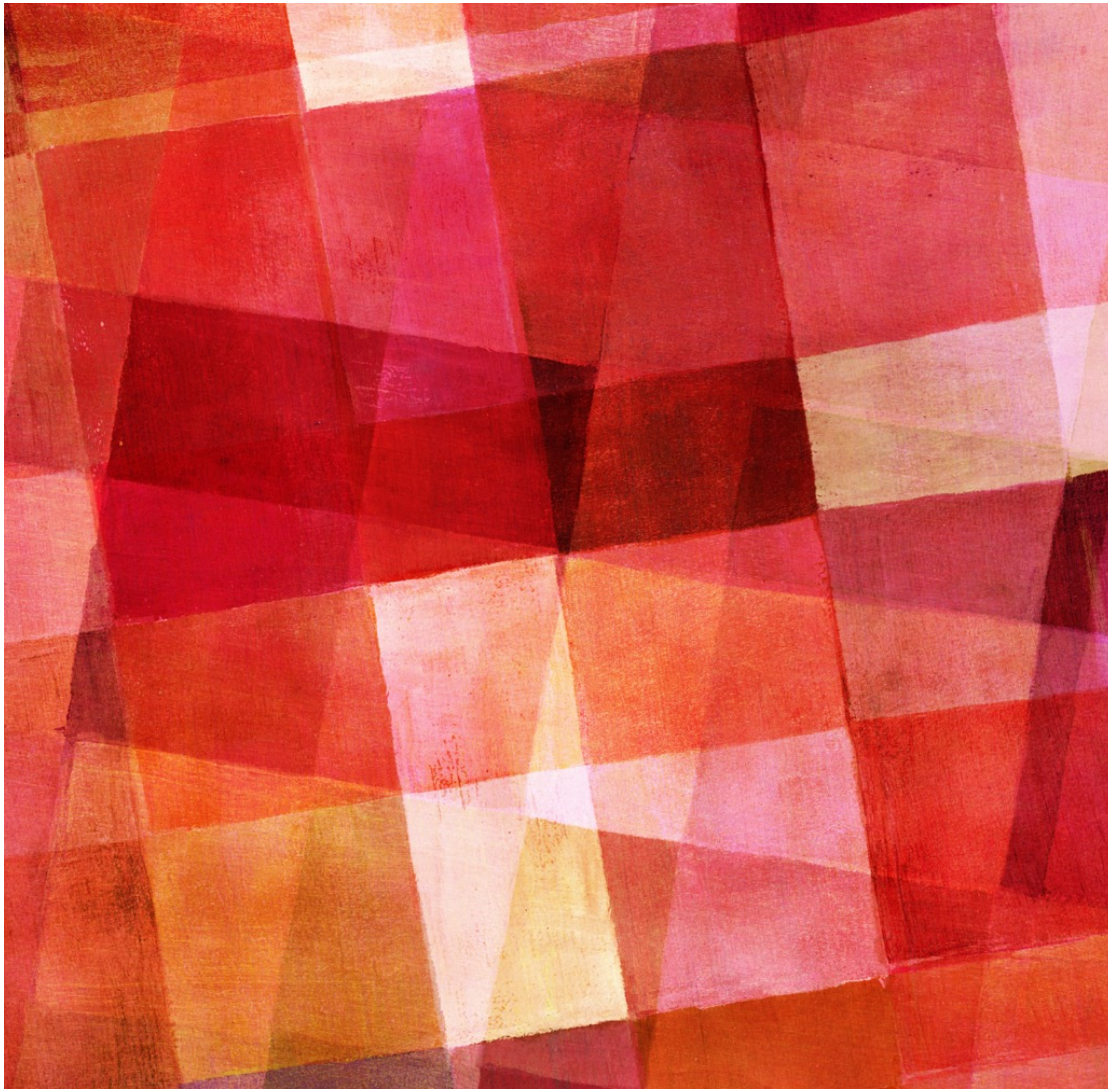

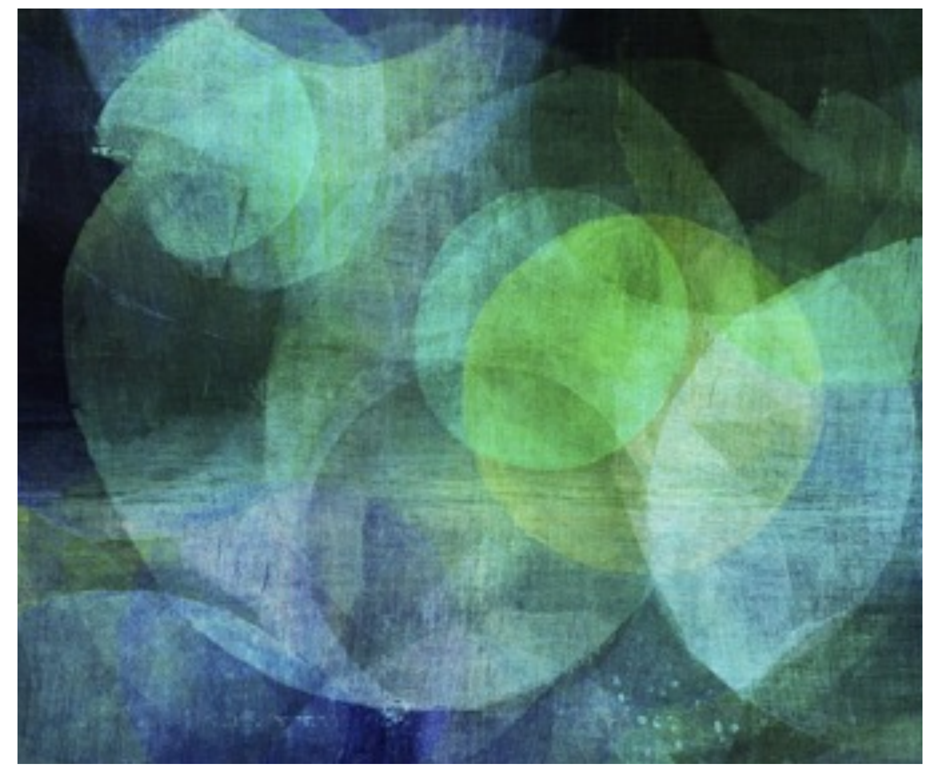

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Reconstruction guarantees

Consider the regularized projection $\mathbf{\Gamma}_n$

$$\mathbf{\Gamma}_n = \Phi_n\Phi_n^{\mathsf{T}}(\Phi_n\Phi_n^{\mathsf{T}} + \gamma\mathbf{I})^{-1} = (\Phi_n\Phi_n^{\mathsf{T}} + \gamma\mathbf{I})^{-1}\Phi_n\Phi_n^{\mathsf{T}}(\Phi_n\Phi_n^{\mathsf{T}} + \gamma\mathbf{I})^{-1}$$
$$= \sum_{i=1}^{n}(\Phi_n\Phi_n^{\mathsf{T}} + \gamma\mathbf{I})^{-1}\phi_i\phi_i^{\mathsf{T}}(\Phi_n\Phi_n^{\mathsf{T}} + \gamma\mathbf{I})^{-1} = \sum_{i=1}^{n}\psi_i\psi_i^{\mathsf{T}}$$
$$\widetilde{\mathbf{\Gamma}}_n = (\Phi_n\Phi_n^{\mathsf{T}} + \gamma\mathbf{I})^{-1}\Phi_n\mathbf{S}_n\mathbf{S}_n^{\mathsf{T}}\Phi_n^{\mathsf{T}}(\Phi_n\Phi_n^{\mathsf{T}} + \gamma\mathbf{I})^{-1} = \sum_{j=1}^{m} w_j\psi_j\psi_j^{\mathsf{T}}$$

An accurate dictionary satisfies

$$\|\mathbf{\Gamma}_n - \widetilde{\mathbf{\Gamma}}_n\|_2^2 \leq \varepsilon$$

equivalent to mixed additive/multiplicative error in quadratic form

$$(1-\varepsilon)\Phi_n\Phi_n^{\mathsf{T}} - \varepsilon\gamma\mathbf{I} \preceq \Phi_n\mathbf{S}_n\mathbf{S}_n^{\mathsf{T}}\Phi_n^{\mathsf{T}} \preceq (1+\varepsilon)\Phi_n\Phi_n^{\mathsf{T}} + \varepsilon\gamma\mathbf{I}$$