Graphs in Machine Learning

Submodularity: Theory

Definition, Properties, and Greedy Algorithm

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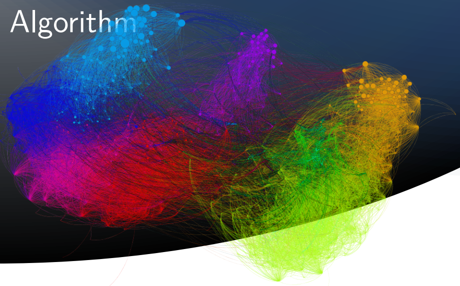

Partially based on material by: Andreas Krause, Branislav Kveton, Michael Kearns

Submodularity: modeling diminishing returns

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A **set function** on a discrete set A is **submodular** if for any $S \subseteq T \subseteq A$ and for any $e \in A \setminus T$

$$f(S \cup \{e\}) - f(S) \geq f(T \cup \{e\}) - f(T)$$

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Link to our **product placement** problem on a **social network graph?**

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Let $S^\star = \arg\max_{S \subseteq A, |S| \leq k} f(S)$ where f is monotonic and submodular set function and let $S_{\texttt{Greedy}}$ be a **greedy solution**.

$$\text{Then} \quad f(S_{\texttt{Greedy}}) \geq \left(1 - \frac{1}{e}\right) \cdot f(S^\star).$$

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Other applications: information, graph cuts, covering, ...

Submodularity: `Greedy` algorithm

```
Input:
    k: the maximum allowed cardinality of the output
    V: a ground set
    f: a monotone, non-negative, and submodular function
Run:
S_0 = ∅
for i = 1 to k do
    S_i ← S_{i-1} ∪ { argmax_{a ∈ V \ S_{i-1}} [f({a} ∪ S_{i-1}) − f(S_{i-1})] }
end for
Output:
    Return S_Greedy = S_k
```

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Submodularity: Approximation guarantee of `Greedy`

Let S_i be the i-th set selected by `Greedy`. We show

$$f\left(S^\star\right) - f\left(S_{i-1}\right) \leq f\left(S^\star \cup S_{i-1}\right) - f\left(S_{i-1}\right)$$

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&\leq f(a \cup S_{i-1})-f(S_{i-1})+f(S^{\star}/a \cup S_{i-1})-f(S_{i-1})
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&\leq \sum_{a \in S^\star \setminus S_{i-1}} \left(f\left(\{a\} \cup S_{i-1}\right) - f\left(S_{i-1}\right)\right)
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& \leq \sum_{a \in S^{\star} \backslash S_{i-1}}\left(f\left(\{a\} \cup S_{i-1}\right)-f\left(S_{i-1}\right)\right) \\
& \leq \sum_{a \in S^{\star} \backslash S_{i-1}}\left(f\left(S_i\right)-f\left(S_{i-1}\right)\right) \leq k\left(f\left(S_i\right)-f\left(S_{i-1}\right)\right)
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Difference from the optimum of $S_{\texttt{Greedy}} = S_k$ after the k-th step

...

$$
\begin{aligned}
f(S^\star) - f(S_k) &= f(S^\star) - f(S_{k-1}) - \left(f\left(S_k\right) - f\left(S_{k-1}\right)\right) \\
&\le f(S^\star) - f(S_{k-1}) - \frac{f(S^\star) - f(S_{k-1})}{k} \\
&\le \left(1 - \tfrac{1}{k}\right) \cdot \left(f(S^\star) - f(S_{k-1})\right) \le \left(1 - \tfrac{1}{k}\right)^k \cdot f(S^\star)
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back to the influence-maximization example ...

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https://misovalko.github.io/mva-ml-graphs.html