# Graphs in Machine Learning

## Online SSL Introduction

Problem Setup and Challenges

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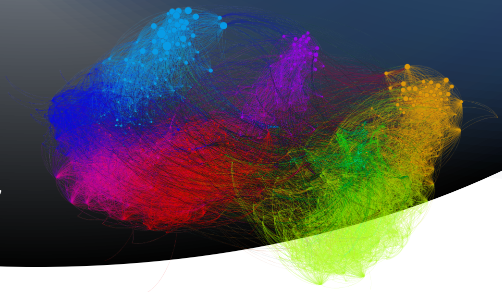
Partially based on material by: Branislav Kveton, Mikhail Belkin, Jerry Zhu



# Online SSL with Graphs

**Offline learning setup**

Given $\{\mathbf{x}_i\}_{i=1}^{N}$ from $\mathbb{R}^d$ and $\{y_i\}_{i=1}^{n_l}$, with $n_l \ll n$, find $\{y_i\}_{i=n_l+1}^{N}$ (**transductive**) or find $f$ predicting $y$ well beyond that (**inductive**).

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What is wrong with this solution?

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**Online HFS: Straightforward solution**

```
while new unlabeled example x_t comes do
    Add x_t to graph G(W)
    Update L_t
    Infer labels
        f_u = (L_uu + γ_g I)^{-1} (W_ul f_l)
    Predict ŷ_t = sgn(f_u(t))
end while
```

where the inference step is

$$\mathbf{f}_u = \left(\mathbf{L}_{uu} + \gamma_g \mathbf{I}\right)^{-1} \left(\mathbf{W}_{ul}\mathbf{f}_l\right)$$

and the prediction is $\widehat{y}_t = \operatorname{sgn}\left(\mathbf{f}_u\left(t\right)\right)$.

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The cost and memory of the operations.

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5: Predict $\widehat{y}_t = \operatorname{sgn}(\mathbf{f}_u(t))$
6: **end while**

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What can we do?

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$$\boldsymbol{\ell}^{\mathrm{q}} = (\mathbf{L}_{uu}^{\mathrm{q}} + \gamma_g V)^{-1} \mathbf{W}_{ul}^{\mathrm{q}} \boldsymbol{\ell}_l \quad \text{where} \quad \mathbf{W}^{\mathrm{q}} = V \widetilde{\mathbf{W}}^{\mathrm{q}} V$$

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Proof?

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Why do we keep the multiplicities?

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**Online HFS with Graph Quantization**

**Input**
    $k$ number of representative nodes

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1: **Input**
2: $k$ number of representative nodes
3: **Initialization**
4: $\mathbf{V}$ matrix of multiplicities with 1 on diagonal

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**Online HFS with Graph Quantization**

**Input**
    $k$ number of representative nodes
**Initialization**
    $\mathbf{V}$ matrix of multiplicities with 1 on diagonal
**while** new unlabeled example $\mathbf{x}_t$ comes **do**
    Add $\mathbf{x}_t$ to graph $G$
    **if** $\#$ nodes $> k$ **then**
        quantize $G$
    **end if**

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    Update $\mathbf{L}_t$ of $G(\mathbf{VWV})$

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    **end if**
    Update $\mathbf{L}_t$ of $G(\mathbf{VWV})$
    Infer labels
    Predict $\widehat{y}_t = \mathrm{sgn}\left(\mathbf{f}_u\left(t\right)\right)$
**end while**

# Michal Valko

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https://misovalko.github.io/mva-ml-graphs.html