# Graphs in Machine Learning

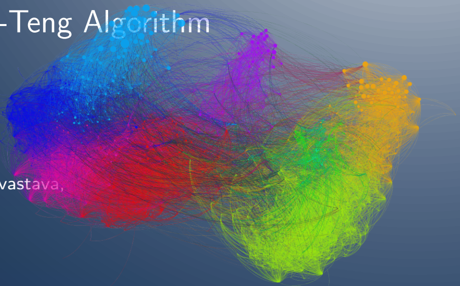
## Spectral Graph Sparsifiers: Theory

Effective Resistance and Spielman-Teng Algorithm

Michal Valko

*Inria & ENS Paris-Saclay, MVA*

Partially based on material by: Rob Fergus, Nikhil Srivastava,
Yiannis Koutis, Joshua Batson, Daniel Spielman



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As a consequence, $\arg\min_{\mathbf{x}} \|\mathbf{L}_H\mathbf{x} - \mathbf{b}\| \approx \arg\min_{\mathbf{x}} \|\mathbf{L}_G\mathbf{x} - \mathbf{b}\|$

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- a single pass over the data

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Laplacian smoothing (denoising): given $\mathbf{y} \triangleq \mathbf{f}^\star + \xi$ and $G$ compute

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Need to approximate spectrum only up to regularization level $\lambda$

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RLA → Graph: Improve over $\mathcal{O}(n \log n)$ exploiting regularization

Graph → RLA: Exploit $\mathbf{L}_G$ structure for fast $(\varepsilon, \gamma)$-sparsification

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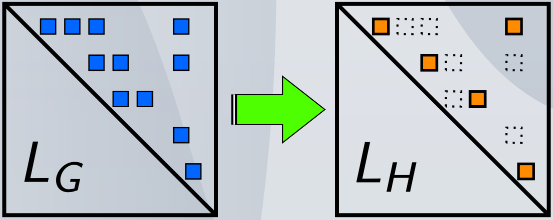


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Then $\sum_{e\in E} s_e\mathbf{v}_e\mathbf{v}_e^\mathsf{T} \approx \mathbf{I} \iff \sum_{e\in E} s_e\mathbf{a}_e\mathbf{a}_e^\mathsf{T} \approx \mathbf{A}$

multiplying by $\mathbf{A}^{1/2}$ on both sides

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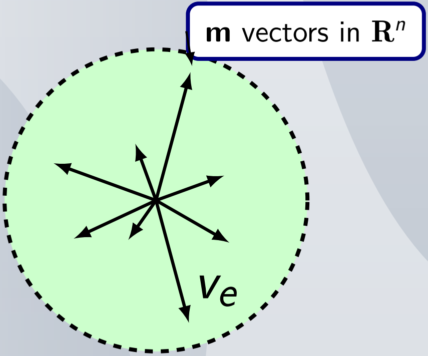


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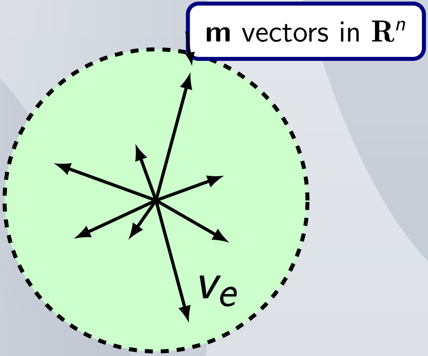


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moment ellipse is a sphere

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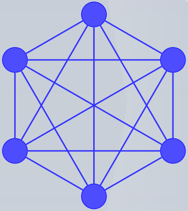

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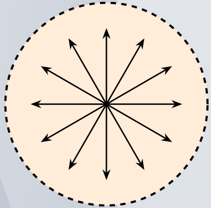

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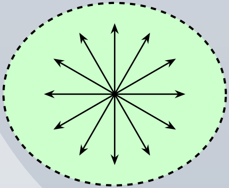

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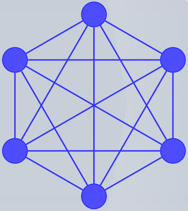

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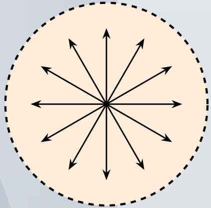

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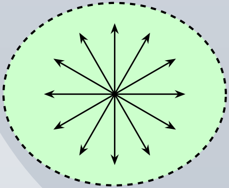
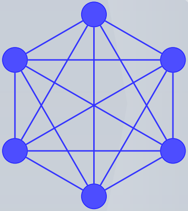


It is already isotropic! (looks like a sphere)

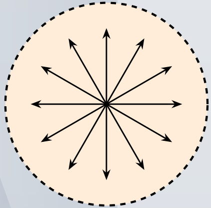
# Spectral Graph Sparsification: Intuition

Example: What happens with $K_n$?

$K_n$ graph



$\sum_{e \in E} \mathbf{b}_e \mathbf{b}_e^\mathsf{T} = \mathbf{L}_G$



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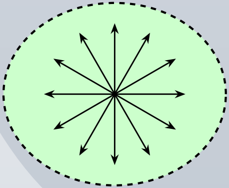

It is already isotropic! (looks like a sphere)

rescaling $\mathbf{v}_e = \mathbf{L}^{-1/2}\mathbf{b}_e$ does not change the shape

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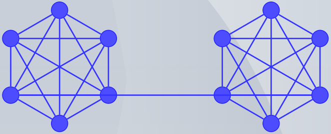
# Spectral Graph Sparsification: Intuition

Example: What happens with a dumbbell?

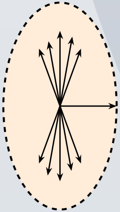
# Spectral Graph Sparsification: Intuition

**Example: What happens with a dumbbell?**

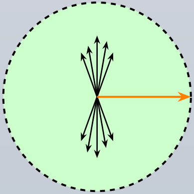
Dumbbell



$\sum_{e \in E} \mathbf{b}_e \mathbf{b}_e^\mathsf{T} = \mathbf{L}_G$



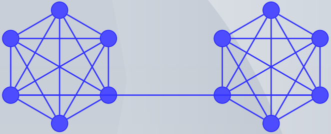
$\sum_{e \in E} \mathbf{v}_e \mathbf{v}_e^\mathsf{T} = \mathbf{I}$



# Spectral Graph Sparsification: Intuition

Example: What happens with a dumbbell?

Dumbbell



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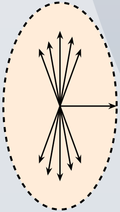

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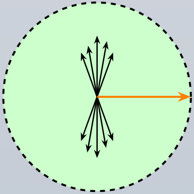
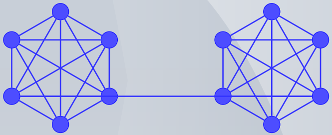


The vector corresponding to the link gets stretched!

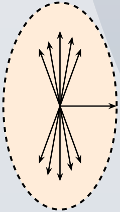
# Spectral Graph Sparsification: Intuition

Example: What happens with a dumbbell?

Dumbbell



$\sum_{e \in E} \mathbf{b}_e \mathbf{b}_e^{\mathsf{T}} = \mathbf{L}_G$



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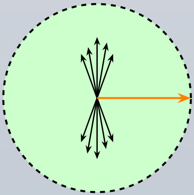


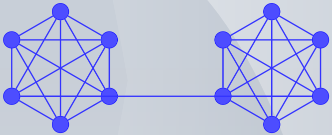
The vector corresponding to the link gets stretched!

because this transformation makes all the directions important

# Spectral Graph Sparsification: Intuition

Example: What happens with a dumbbell?

Dumbbell



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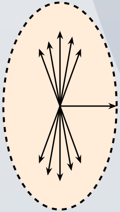

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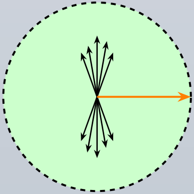

The vector corresponding to the link gets stretched!

because this transformation makes all the directions important

rescaling reveals the vectors that are critical

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# References I


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# Michal Valko

michal.valko@inria.fr

Inria & ENS Paris-Saclay, MVA


https://misovalko.github.io/mva-ml-graphs.html