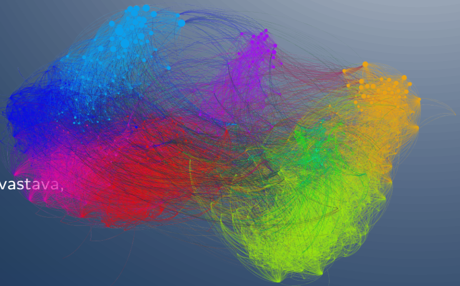
Graphs in Machine Learning

Spectral Graph Sparsifiers in ML

Ridge Sparsifiers and Applications

Michal Valko

Inria & ENS Paris-Saclay, MVA



Partially based on material by: Rob Fergus, Nikhil Srivastava,
Yiannis Koutis, Joshua Batson, Daniel Spielman

Spectral Graph Sparsification: Intuition

What it this rescaling $\mathbf{v}_e = \mathbf{L}_G^{-1/2}\mathbf{b}_e$ doing to the norm?

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Random walk intuition: inverse of number of alternative paths

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Edges with higher R_{eff} are more **electrically significant**!

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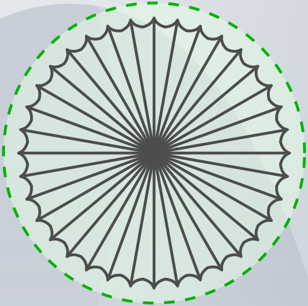
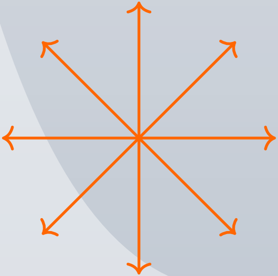


m vectors in $\mathbb{R}^n$



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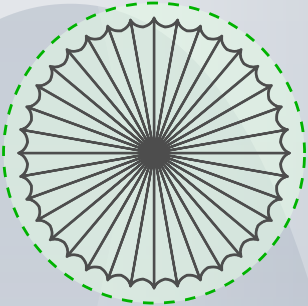

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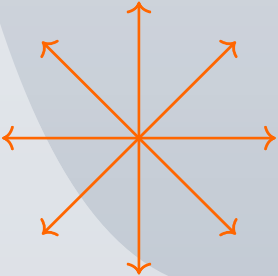

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We take a subset of these $\mathbf{e}_e$s and scale them!

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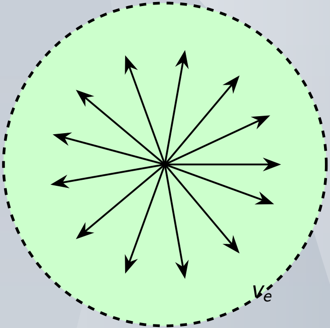

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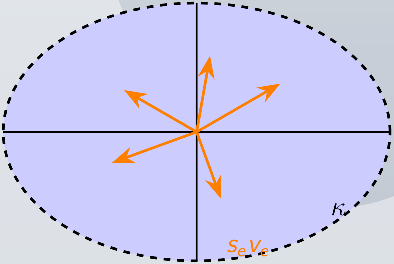

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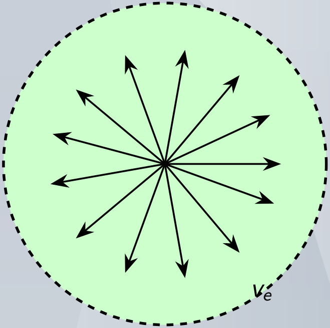

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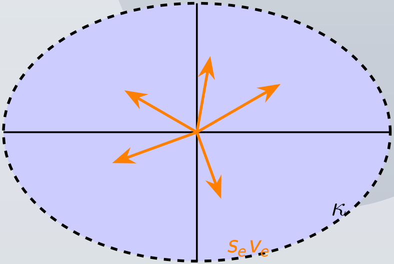

Such that the blue ellipsoid looks like identity!

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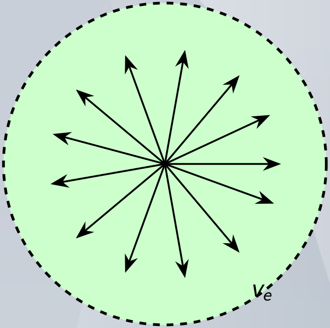

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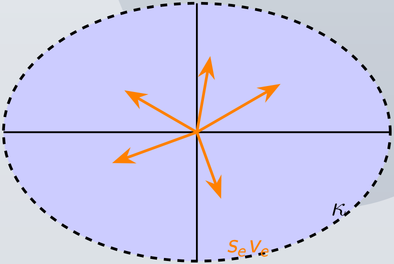

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the blue eigenvalues are between 1 and κ

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finer bounds now available

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Definition

γ-effective resistance: $r_e(\gamma) = \mathbf{b}_e^\mathsf{T}(\mathbf{L}_G + \gamma\mathbf{I})^{-1}\mathbf{b}_e$

Effective dim.: $\mathrm{d}_{\text{eff}}(\gamma) = \sum_e r_e(\gamma) = \sum_{i=1}^{n} \frac{\lambda_i(\mathbf{L}_G)}{\lambda_i(\mathbf{L}_G)+\gamma} \leq n$

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Adapt SOA algorithm for kernel matrix approximation

SQUEAK, Calandriello et al., 2017

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- Fast solvers for SDD systems:
  - ↳ use sparsification internally

all the way until you hit the turtles

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Chicken and egg problem

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Sampling according to approximate effective resistances
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Use $\widetilde{R}_{\text{eff}}$ to compute an improved approximate sparsifier H ↰
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Computing $\widetilde{R}_{\text{eff}}$ using the sparsifier is fast ($m = \boldsymbol{O}(n\log(n))$), and not too many iterations are necessary.

What can I use sparsifiers for?

- Graph linear systems: minimum cut, maximum flow, Laplacian regression, SSL

But what if my problems have no use for spectral guarantees?

Or if my boss does not trust approximation methods

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- Various embeddings: k-means, spectral clustering.

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Inria & ENS Paris-Saclay, MVA

https://misovalko.github.io/mva-ml-graphs.html