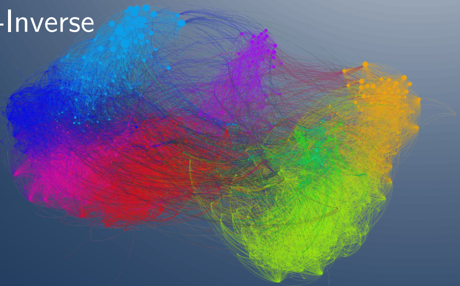
Graphs in Machine Learning

Effective Resistance Computation

Laplacian Connection and Pseudo-Inverse

Michal Valko

Inria & ENS Paris-Saclay, MVA



Partially based on material by: Ulrike von Luxburg, Gary Miller, Doyle & Schnell, Daniel Spielman

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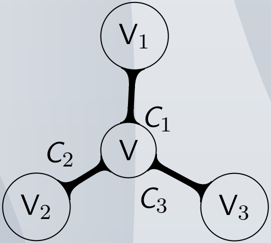

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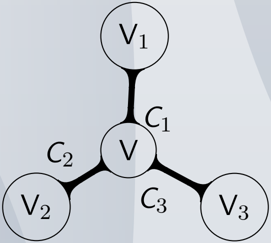

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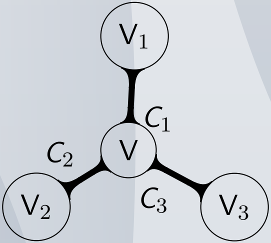

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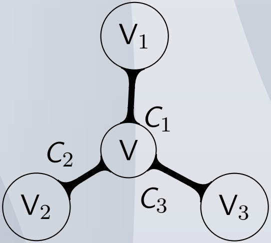

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Kirchhoff says: This is zero! **There is no residual current!**

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The net injected has to be zero $\equiv$ Kirchhoff's Law.

Resistors and the Laplacian: Finding R_{ab}

Let's calculate R_{1N} to get the **movie recommendation score**!

$$\mathbf{L}\begin{pmatrix} 0 \\ v_2 \\ \vdots \\ v_{n-1} \\ 1 \end{pmatrix} = \begin{pmatrix} i \\ 0 \\ \vdots \\ 0 \\ -i \end{pmatrix}$$

$$i = \frac{V}{R} \qquad V = 1 \qquad R = \frac{1}{i}$$

Return $R_{1N} = \dfrac{1}{i}$

Doyle and Snell: Random Walks and Electric Networks

https://math.dartmouth.edu/~doyle/docs/walks/walks.pdf

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V_i is a **convex combination of its neighbors**

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Notice: We can reuse $\mathbf{L}^{+}$ to get resistances for any pair of nodes!

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https://misovalko.github.io/mva-ml-graphs.html