Graphs in Machine Learning

Normalized Laplacians

L_{sym} and L_{rw}

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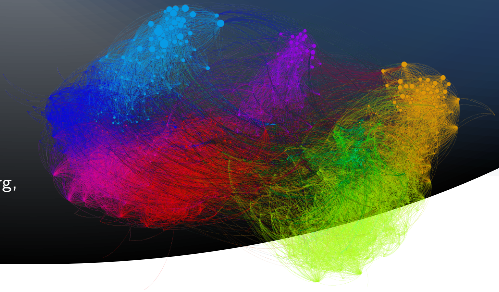
Inria & ENS Paris-Saclay, MVA

Partially based on material by: Ulrike von Luxburg,
Miller, Doyle & Schnell, Daniel Spielman



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$$S_G(\mathbf{f}) = \mathbf{f}^\mathsf{T}\mathbf{L}\mathbf{f} = \mathbf{f}^\mathsf{T}\mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^\mathsf{T}\mathbf{f} = \boldsymbol{\alpha}^\mathsf{T}\boldsymbol{\Lambda}\boldsymbol{\alpha} = \|\boldsymbol{\alpha}\|_{\boldsymbol{\Lambda}}^2 = \sum_{i=1}^{N}\lambda_i\alpha_i^2$$

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The smoothness of k-th eigenvector is the k-th eigenvalue.

Laplacian of the Complete Graph K_N

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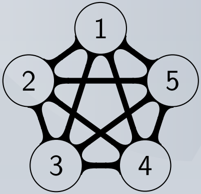

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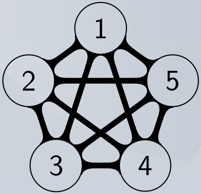

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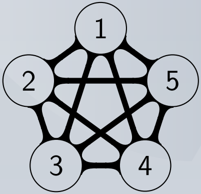

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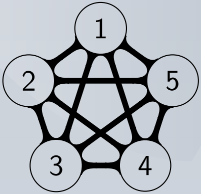

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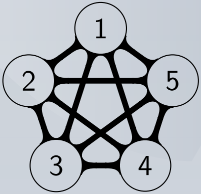

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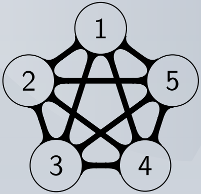

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Answer: $N-1$ eigenvectors $\perp \mathbf{1}_N$ for eigenvalue N with multiplicity $N-1$.

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$(\lambda, \mathbf{u})$ is an eigenpair for $\mathbf{L}_{rw}$ iff $(\lambda, \mathbf{D}^{1/2}\mathbf{u})$ is an eigenpair for $\mathbf{L}_{sym}$

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https://misovalko.github.io/mva-ml-graphs.html