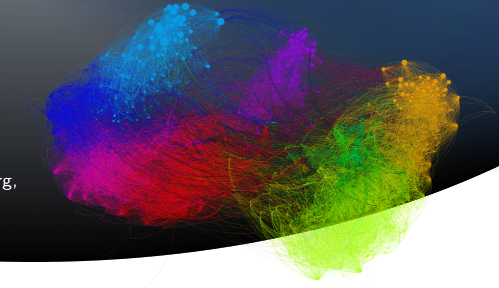
Graphs in Machine Learning

Laplacian and Random Walks

Stationary Distribution

Michal Valko

Inria & ENS Paris-Saclay, MVA



Partially based on material by: Ulrike von Luxburg, Miller, Doyle & Schnell, Daniel Spielman

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$\mathbf{L}_{sym}$ and $\mathbf{L}_{rw}$ are PSD with non-negative real eigenvalues

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https://misovalko.github.io/mva-ml-graphs.html