# Graphs in Machine Learning

## Analysis of Online SSL

Quantization Error and Performance Analysis

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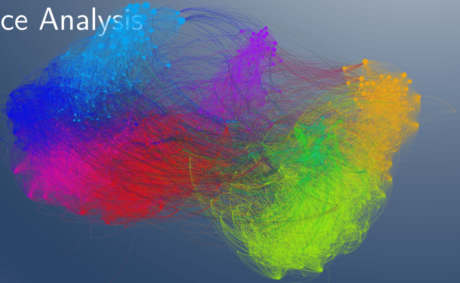
Partially based on material by: Branislav Kveton, Mikhail Belkin, Jerry Zhu



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- online error — data only incrementally: $(\hat{f}_{so,t}[t] - \hat{f}_{s,t})^2$
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All together:

$$\frac{1}{N}\sum_{t=1}^{N}(\hat{f}_{soq,t}[t]-y_t)^2 \leq \frac{9}{2N}\sum_{t=1}^{N}(\hat{f}_{s,t}-y_t)^2+\frac{9}{2N}\sum_{t=1}^{N}(\hat{f}_{so,t}[t]-\hat{f}_{s,t})^2+\frac{9}{2N}\sum_{t=1}^{N}(\hat{f}_{soq,t}[$$

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If all labeled examples $l$ are i.i.d., $c_l = 1$ and $c_l \gg c_u$, then

$$R(\boldsymbol{\ell}^\star) \leq \widehat{R}(\boldsymbol{\ell}^\star) + \underbrace{\beta + \sqrt{\frac{2\ln(2/\delta)}{n_l}}(n_l\beta + 4)}_{\text{transductive error } \Delta_T(\beta, n_l, \delta)}$$

$$\beta \leq 2\left[\frac{\sqrt{2}}{\gamma_g + 1} + \sqrt{2n_l}\frac{1 - c_u}{c_u}\frac{\lambda_M(\mathbf{L}) + \gamma_g}{\gamma_g^2 + 1}\right]$$

holds with the probability of $1 - \delta$, where

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Idea: If $\mathbf{L}$ and $\hat{L}_o$ are regularized, then HFSs get closer together.

since they get closer to zero

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In $\mathbf{Q}$! $\hat{K}_o$ (online) vs. $\tilde{K}$ (quantized)

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In $\mathbf{Q}$! $\hat{K}_o$ (online) vs. $\tilde{K}$ (quantized)

We have: $\hat{\ell}_o = (\mathbf{C}^{-1} \hat{K}_o + \mathbf{I})^{-1} \mathbf{y}$ vs. $\hat{\ell}_{oq} = (\mathbf{C}^{-1} \tilde{K} + \mathbf{I})^{-1} \mathbf{y}$

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With linear algebra we get

$$\|\hat{\ell}_{oq} - \hat{\ell}_o\|_2 \leq \frac{\sqrt{n_l}}{c_u \gamma_g^2} \|\hat{K}_{oq} - \hat{K}_o\|_F$$

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When can we keep $\|\hat{L}_{oq} - \hat{L}_o\|_F$ under control?

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Charikar guarantees **distortion** error of at most $RR/(R-1)$

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**Bounding** **quantization error** $\left(\hat{f}_{soq,t}[t] - \hat{f}_{so,t}[t]\right)^2$

The quantization error depends on $\|\hat{K}_{oq} - \hat{K}_o\|_F = \|\hat{L}_{oq} - \hat{L}_o\|_F$.

When can we keep $\|\hat{L}_{oq} - \hat{L}_o\|_F$ under control?

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Assume manifold $\mathcal{M}$

- all $\{\mathbf{x}_i\}_{i\geq 1}$ lie on a smooth $d$-dimensional compact $\mathcal{M}$
- with boundary of bounded geometry Def. 11 of Hein [**hein2007graph**]
  - has finite volume $V$
  - has finite surface area $A$
  - should not intersect itself
  - should not fold back onto itself

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Bounding $\|\hat{L}_{oq} - \hat{L}_o\|_F$ when $\mathbf{x}_i \in \mathcal{M}$

Consider $k$-sphere packing$^\star$ of radius $r$ with centers contained in $\mathcal{M}$. $^\star$only the centers are packed, not necessarily the entire ball

If $k$ is large $\rightarrow r <$ **injectivity radius** of $\mathcal{M}$ [**hein2007graph**]

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If $k$ is large $\rightarrow r <$ **injectivity radius** of $\mathcal{M}$ [**hein2007graph**] and $r < 1$:

$$r < \left(\frac{V + Ac_{\mathcal{M}}}{kc_d}\right)^{1/d} = \mathcal{O}\left(\frac{1}{k^{1/d}}\right)$$

$r$-packing is a $2r$-covering:

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$r$-packing is a $2r$-covering:

$$\max_{i=1,\dots,N} \|\mathbf{x}_i - \mathbf{c}\|_2 \leq R\frac{R}{R-1} \leq 2\mathcal{O}\left(k^{-1/d}\right) = \mathcal{O}\left(k^{-1/d}\right)$$

But what about $\|\hat{L}_{oq} - \hat{L}_o\|_F$?

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Are the assumptions reasonable?

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$$\frac{1}{N}\sum_{t=1}^{N}(\hat{f}_{soq,t}[t] - \hat{f}_{so,t}[t])^2 \le \frac{n_l}{c_u^2\gamma_g^4}\|\hat{L}_{oq} - \hat{L}_o\|_F^2$$

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What does that mean?

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https://misovalko.github.io/mva-ml-graphs.html