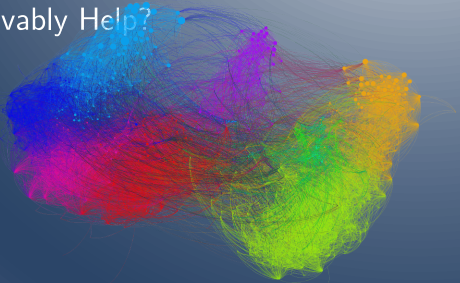
Graphs in Machine Learning

SSL Learnability

When Does Graph-Based SSL Provably Help?



Michal Valko

Inria & ENS Paris-Saclay, MVA

Partially based on material by: Branislav Kveton, Mikhail Belkin, Jerry Zhu

SSL with Graphs: What is behind it?

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http://people.cs.uchicago.edu/~niyogi/papersps/ssminimax2.pdf

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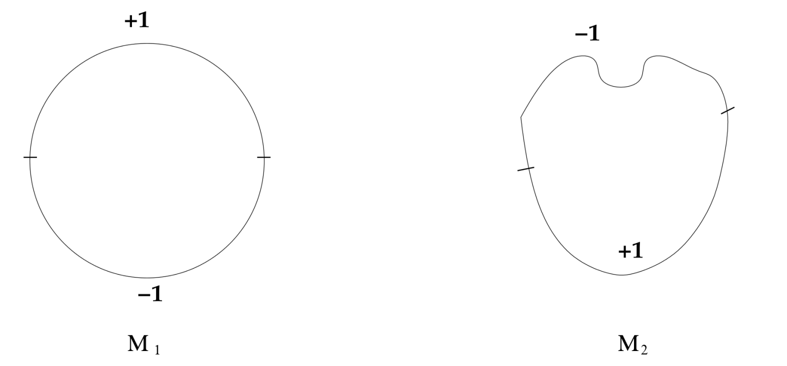

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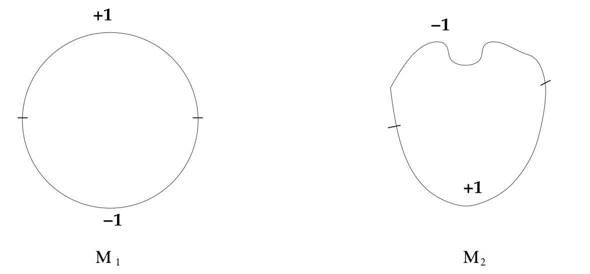
In which cases there is a gap between $Q(n_l, \mathcal{P})$ and $R(n_l, \mathcal{P})$?

SSL with Graphs: What is behind it?

Hypothesis space $\mathcal{H}$: half of the circle as $+1$ and the rest as -1



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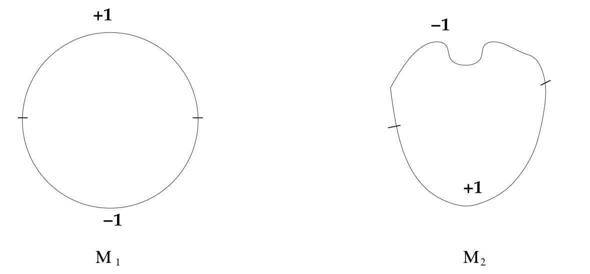

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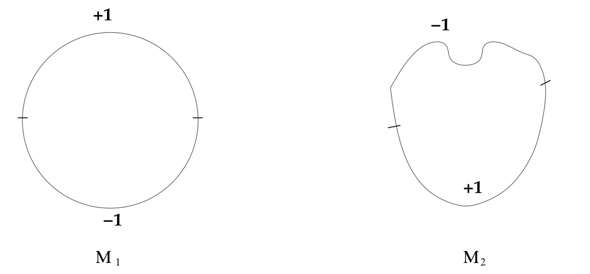

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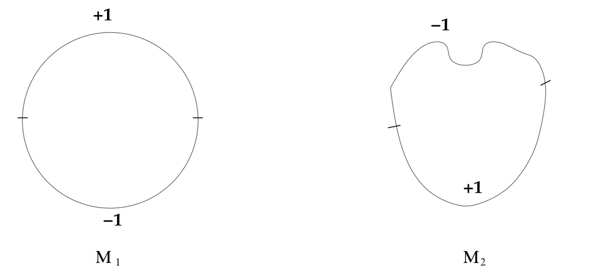

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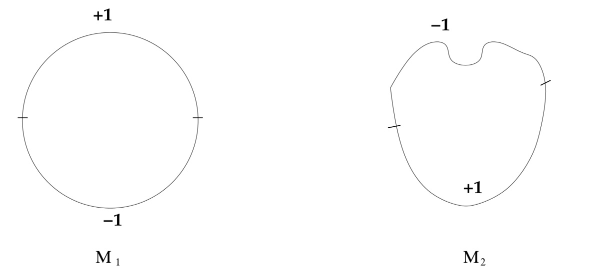

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$$\text{Optimal rate } Q(n, \mathcal{P}) \leq 2\sqrt{\frac{3 \log n_l}{n_l}}$$

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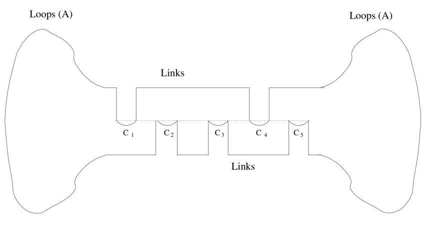
$$\mathcal{M} = \text{Loops} \cup \text{Links} \cup C \text{ where } C = \cup_{i=1}^d C_i$$



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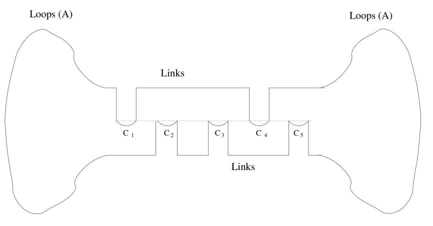

Main idea: d segments in C, $d - l$ with no data, 2^l possible choices for labels, which helps us to lower bound $\|A(\bar{z}) - m_p\|_{L^2(p_X)}$

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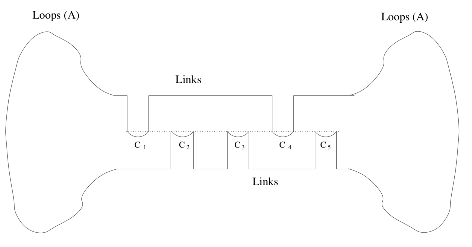

Knowing the manifold helps

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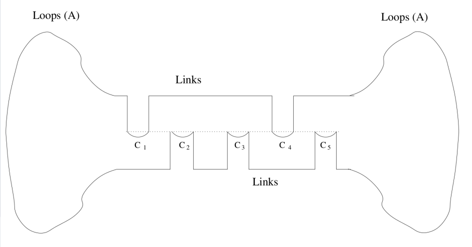

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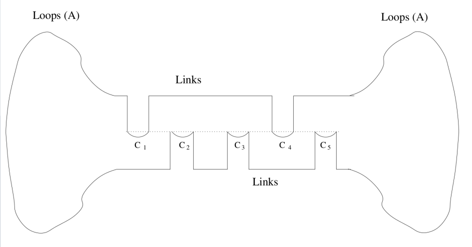

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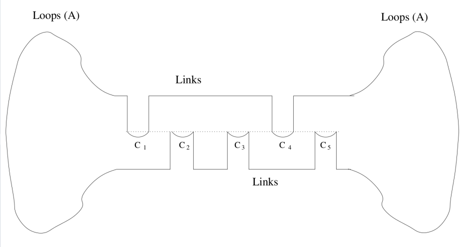

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- altogether: **closeness on manifold** $\rightarrow$ **similarity in labels**

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What does it mean to **know** $\mathcal{M}$?

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What does it mean to **know** $\mathcal{M}$?

Different degrees of knowing $\mathcal{M}$

- set membership oracle: $\mathbf{x} \stackrel{?}{\in} \mathcal{M}$
- approximate oracle
- knowing the harmonic functions on $\mathcal{M}$
- knowing the Laplacian $\mathcal{L}_{\mathcal{M}}$
- knowing eigenvalues and eigen*functions*
- topological invariants, e.g., dimension
- metric information: geodesic distance

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https://misovalko.github.io/mva-ml-graphs