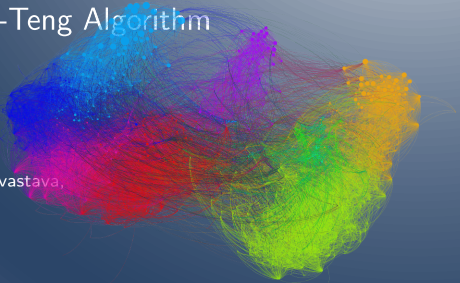
Graphs in Machine Learning

Spectral Graph Sparsifiers: Theory

Effective Resistance and Spielman-Teng Algorithm



Michal Valko

Inria & ENS Paris-Saclay, MVA

Partially based on material by: Rob Fergus, Nikhil Srivastava, Yiannis Koutis, Joshua Batson, Daniel Spielman

Spectral Graph Sparsifiers

Rayleigh-Ritz gives:

Spectral Graph Sparsifiers

Rayleigh-Ritz gives:

$$\lambda_{\min} = \min \frac{\mathbf{x}^\mathsf{T}\mathbf{L}\mathbf{x}}{\mathbf{x}^\mathsf{T}\mathbf{x}} \quad \text{and} \quad \lambda_{\max} = \max \frac{\mathbf{x}^\mathsf{T}\mathbf{L}\mathbf{x}}{\mathbf{x}^\mathsf{T}\mathbf{x}}$$

Spectral Graph Sparsifiers

Rayleigh-Ritz gives:

$$\lambda_{\min} = \min \frac{\mathbf{x}^\mathsf{T}\mathbf{L}\mathbf{x}}{\mathbf{x}^\mathsf{T}\mathbf{x}} \quad \text{and} \quad \lambda_{\max} = \max \frac{\mathbf{x}^\mathsf{T}\mathbf{L}\mathbf{x}}{\mathbf{x}^\mathsf{T}\mathbf{x}}$$

What can we say about $\lambda_i(G)$ and $\lambda_i(H)$?

Spectral Graph Sparsifiers

Rayleigh-Ritz gives:

$$\lambda_{\min} = \min \frac{\mathbf{x}^\mathsf{T}\mathbf{L}\mathbf{x}}{\mathbf{x}^\mathsf{T}\mathbf{x}} \quad \text{and} \quad \lambda_{\max} = \max \frac{\mathbf{x}^\mathsf{T}\mathbf{L}\mathbf{x}}{\mathbf{x}^\mathsf{T}\mathbf{x}}$$

What can we say about $\lambda_i(G)$ and $\lambda_i(H)$?

$$(1-\varepsilon)\mathbf{f}^\mathsf{T}\mathbf{L}_G\mathbf{f} \leq \mathbf{f}^\mathsf{T}\mathbf{L}_H\mathbf{f} \leq (1+\varepsilon)\mathbf{f}^\mathsf{T}\mathbf{L}_G\mathbf{f}$$

Spectral Graph Sparsifiers

Rayleigh-Ritz gives:

$$\lambda_{\min} = \min \frac{\mathbf{x}^\mathsf{T}\mathbf{L}\mathbf{x}}{\mathbf{x}^\mathsf{T}\mathbf{x}} \quad \text{and} \quad \lambda_{\max} = \max \frac{\mathbf{x}^\mathsf{T}\mathbf{L}\mathbf{x}}{\mathbf{x}^\mathsf{T}\mathbf{x}}$$

What can we say about $\lambda_i(G)$ and $\lambda_i(H)$?

$$(1-\varepsilon)\mathbf{f}^\mathsf{T}\mathbf{L}_G\mathbf{f} \le \mathbf{f}^\mathsf{T}\mathbf{L}_H\mathbf{f} \le (1+\varepsilon)\mathbf{f}^\mathsf{T}\mathbf{L}_G\mathbf{f}$$

Eigenvalues are approximated well!

$$(1-\varepsilon)\lambda_i(G) \le \lambda_i(H) \le (1+\varepsilon)\lambda_i(G)$$

Using matrix ordering notation $(1-\varepsilon)\mathbf{L}_G \preceq \mathbf{L}_H \preceq (1+\varepsilon)\mathbf{L}_G$

Spectral Graph Sparsifiers

Rayleigh-Ritz gives:

$$\lambda_{\min} = \min \frac{\mathbf{x}^\mathsf{T}\mathbf{L}\mathbf{x}}{\mathbf{x}^\mathsf{T}\mathbf{x}} \quad \text{and} \quad \lambda_{\max} = \max \frac{\mathbf{x}^\mathsf{T}\mathbf{L}\mathbf{x}}{\mathbf{x}^\mathsf{T}\mathbf{x}}$$

What can we say about $\lambda_i(G)$ and $\lambda_i(H)$?

$$(1-\varepsilon)\mathbf{f}^\mathsf{T}\mathbf{L}_G\mathbf{f} \le \mathbf{f}^\mathsf{T}\mathbf{L}_H\mathbf{f} \le (1+\varepsilon)\mathbf{f}^\mathsf{T}\mathbf{L}_G\mathbf{f}$$

Eigenvalues are approximated well!

$$(1-\varepsilon)\lambda_i(G) \le \lambda_i(H) \le (1+\varepsilon)\lambda_i(G)$$

Using matrix ordering notation $(1-\varepsilon)\mathbf{L}_G \preceq \mathbf{L}_H \preceq (1+\varepsilon)\mathbf{L}_G$

As a consequence, $\arg\min_\mathbf{x} \|\mathbf{L}_H\mathbf{x} - \mathbf{b}\| \approx \arg\min_\mathbf{x} \|\mathbf{L}_G\mathbf{x} - \mathbf{b}\|$

Spectral Graph Sparsifiers in ML

Proposition ([**spielman2011graph**; **kyng2016framework**])

There exists an algorithm that can construct a spectral ε-sparsifier

Spectral Graph Sparsifiers in ML

Proposition ([spielman2011graph; kyng2016framework])

There exists an algorithm that can construct a spectral ε-sparsifier

- with only $\mathcal{O}(n \log(n)/\varepsilon^2)$ edges

Spectral Graph Sparsifiers in ML

Proposition ([spielman2011graph; kyng2016framework])

There exists an algorithm that can construct a spectral ε-sparsifier

- with only $\mathcal{O}(n \log(n)/\varepsilon^2)$ edges
- in $\mathcal{O}(m \log^2(n))$ time and $\mathcal{O}(n \log(n)/\varepsilon^2)$ space

Spectral Graph Sparsifiers in ML

Proposition ([spielman2011graph; kyng2016framework])

There exists an algorithm that can construct a spectral ε-sparsifier

- with only $\mathcal{O}(n \log(n)/\varepsilon^2)$ edges
- in $\mathcal{O}(m \log^2(n))$ time and $\mathcal{O}(n \log(n)/\varepsilon^2)$ space
- a single pass over the data

Spectral Graph Sparsifiers in ML

Laplacian smoothing (denoising): given $\mathbf{y} \triangleq \mathbf{f}^\star + \xi$ and G compute

$$\min_{\mathbf{f} \in \mathbb{R}^n} (\mathbf{f} - \mathbf{y})^\mathsf{T} (\mathbf{f} - \mathbf{y}) + \lambda \mathbf{f}^\mathsf{T} \mathbf{L}_G \mathbf{f} \qquad (1)$$

Spectral Graph Sparsifiers in ML

Laplacian smoothing (denoising): given $\mathbf{y} \triangleq \mathbf{f}^\star + \xi$ and G compute

$$\min_{\mathbf{f}\in\mathbb{R}^n} (\mathbf{f}-\mathbf{y})^\mathsf{T}(\mathbf{f}-\mathbf{y}) + \lambda \mathbf{f}^\mathsf{T}\mathbf{L}_G\mathbf{f} \tag{1}$$

	Preproc	Time	Space
$\widehat{\mathbf{f}} = (\lambda\mathbf{L}_G + \mathbf{I})^{-1}\mathbf{y}$	0	$\mathcal{O}(m\log(n))$	$\mathcal{O}(m)$

Spectral Graph Sparsifiers in ML

Laplacian smoothing (denoising): given $\mathbf{y} \triangleq \mathbf{f}^\star + \xi$ and G compute

$$\min_{\mathbf{f}\in\mathbb{R}^n} (\mathbf{f}-\mathbf{y})^\mathsf{T}(\mathbf{f}-\mathbf{y}) + \lambda \mathbf{f}^\mathsf{T}\mathbf{L}_G\mathbf{f} \tag{1}$$

	Preproc	Time	Space
$\widehat{\mathbf{f}} = (\lambda\mathbf{L}_G + \mathbf{I})^{-1}\mathbf{y}$	0	$\mathcal{O}(m\log(n))$	$\mathcal{O}(m)$
$\widetilde{\mathbf{f}} = (\lambda\mathbf{L}_H + \mathbf{I})^{-1}\mathbf{y}$	$\mathcal{O}(m\log^2(n))$	$\mathcal{O}(n\log^2(n))$	$\mathcal{O}(n\log(n))$

Spectral Graph Sparsifiers in ML

Laplacian smoothing (denoising): given $\mathbf{y} \triangleq \mathbf{f}^\star + \xi$ and G compute

$$\min_{\mathbf{f}\in\mathbb{R}^n} (\mathbf{f}-\mathbf{y})^\mathsf{T}(\mathbf{f}-\mathbf{y}) + \lambda \mathbf{f}^\mathsf{T}\mathbf{L}_G\mathbf{f} \quad (1)$$

	Preproc	Time	Space
$\widehat{\mathbf{f}} = (\lambda\mathbf{L}_G + \mathbf{I})^{-1}\mathbf{y}$	0	$\mathcal{O}(m\log(n))$	$\mathcal{O}(m)$
$\widetilde{\mathbf{f}} = (\lambda\mathbf{L}_H + \mathbf{I})^{-1}\mathbf{y}$	$\mathcal{O}(m\log^2(n))$	$\mathcal{O}(n\log^2(n))$	$\mathcal{O}(n\log(n))$

Large computational improvement
↳ accuracy guarantees! [sadhanala2016graph]

Spectral Graph Sparsifiers in ML

Laplacian smoothing (denoising): given $\mathbf{y} \triangleq \mathbf{f}^{\star} + \xi$ and G compute

$$\min_{\mathbf{f} \in \mathbb{R}^n} (\mathbf{f} - \mathbf{y})^{\mathsf{T}} (\mathbf{f} - \mathbf{y}) + \lambda \mathbf{f}^{\mathsf{T}} \mathbf{L}_G \mathbf{f} \qquad (1)$$

	Preproc	Time	Space
$\widehat{\mathbf{f}} = (\lambda \mathbf{L}_G + \mathbf{I})^{-1} \mathbf{y}$	0	$\mathcal{O}(m \log(n))$	$\mathcal{O}(m)$
$\widetilde{\mathbf{f}} = (\lambda \mathbf{L}_H + \mathbf{I})^{-1} \mathbf{y}$	$\mathcal{O}(m \log^2(n))$	$\mathcal{O}(n \log^2(n))$	$\mathcal{O}(n \log(n))$

Large computational improvement
↳ accuracy guarantees! [sadhanala2016graph]

Need to approximate spectrum only up to regularization level λ

Ridge Spectral Graph Sparsifiers in ML

Definition

An (ε, γ)-sparsifier of G is a reweighted subgraph H s.t.

$$(1 - \varepsilon)\mathbf{L}_G - \varepsilon\gamma\mathbf{I} \preceq \mathbf{L}_H \preceq (1 + \varepsilon)\mathbf{L}_G + \varepsilon\gamma\mathbf{I} \quad (2)$$

Ridge Spectral Graph Sparsifiers in ML

Definition

An (ε, γ)-sparsifier of G is a reweighted subgraph H s.t.

$$(1-\varepsilon)\mathbf{L}_G - \varepsilon\gamma\mathbf{I} \preceq \mathbf{L}_H \preceq (1+\varepsilon)\mathbf{L}_G + \varepsilon\gamma\mathbf{I} \tag{2}$$

Mixed multiplicative/additive error

- large (i.e. $\geq \gamma$) directions reconstructed accurately
- small (i.e. $\leq \gamma$) directions uniformly approximated ($\gamma\mathbf{I}$)

$$(1-\varepsilon)\mathbf{L}_G - \varepsilon\gamma\mathbf{I} \preceq \mathbf{L}_H \preceq (1+\varepsilon)\mathbf{L}_G + \varepsilon\gamma\mathbf{I} \quad (2)$$

Mixed multiplicative/additive error

- large (i.e. $\geq \gamma$) directions reconstructed accurately
- small (i.e. $\leq \gamma$) directions uniformly approximated ($\gamma\mathbf{I}$)

Adapted from Randomized Linear Algebra (RLA) community
↳ PSD matrix low-rank approx. [alaoui2014fast]

Ridge Spectral Graph Sparsifiers in ML

Definition

An (ε, γ)-sparsifier of G is a reweighted subgraph H s.t.

$$(1-\varepsilon)\mathbf{L}_G - \varepsilon\gamma\mathbf{I} \preceq \mathbf{L}_H \preceq (1+\varepsilon)\mathbf{L}_G + \varepsilon\gamma\mathbf{I} \quad (2)$$

Mixed multiplicative/additive error

- large (i.e. $\geq \gamma$) directions reconstructed accurately
- small (i.e. $\leq \gamma$) directions uniformly approximated ($\gamma\mathbf{I}$)

Adapted from Randomized Linear Algebra (RLA) community
↳ PSD matrix low-rank approx. [alaoui2014fast]

RLA → Graph: Improve over $\mathcal{O}(n \log n)$ exploiting regularization

Graph → RLA: Exploit $\mathbf{L}_G$ structure for fast (ε, γ)-sparsification

Spectral Graph Sparsification: Intuition

Let us consider unweighted graphs: $w_{ij} \in \{0, 1\}$

$$\mathbf{L}_G = \sum_{ij} w_{ij} \mathbf{L}_{ij} = \sum_{ij \in E} \mathbf{L}_{ij}$$

Spectral Graph Sparsification: Intuition

Let us consider unweighted graphs: $w_{ij} \in \{0, 1\}$

$$\mathbf{L}_G = \sum_{ij} w_{ij} \mathbf{L}_{ij} = \sum_{ij \in E} \mathbf{L}_{ij} = \sum_{ij \in E} (\boldsymbol{\delta}_i - \boldsymbol{\delta}_j)(\boldsymbol{\delta}_i - \boldsymbol{\delta}_j)^\mathsf{T}$$

Spectral Graph Sparsification: Intuition

Let us consider unweighted graphs: $w_{ij} \in \{0, 1\}$

$$\mathbf{L}_G = \sum_{ij} w_{ij}\mathbf{L}_{ij} = \sum_{ij \in E} \mathbf{L}_{ij} = \sum_{ij \in E} (\boldsymbol{\delta}_i - \boldsymbol{\delta}_j)(\boldsymbol{\delta}_i - \boldsymbol{\delta}_j)^\mathsf{T} = \sum_{e \in E} \mathbf{b}_e \mathbf{b}_e^\mathsf{T}$$

Spectral Graph Sparsification: Intuition

Let us consider unweighted graphs: $w_{ij} \in \{0, 1\}$

$$\mathbf{L}_G = \sum_{ij} w_{ij}\mathbf{L}_{ij} = \sum_{ij \in E} \mathbf{L}_{ij} = \sum_{ij \in E} (\boldsymbol{\delta}_i - \boldsymbol{\delta}_j)(\boldsymbol{\delta}_i - \boldsymbol{\delta}_j)^\mathsf{T} = \sum_{e \in E} \mathbf{b}_e \mathbf{b}_e^\mathsf{T}$$

We look for a **subgraph** H

$$\mathbf{L}_H = \sum_{e \in E} s_e \mathbf{b}_e \mathbf{b}_e^\mathsf{T}$$

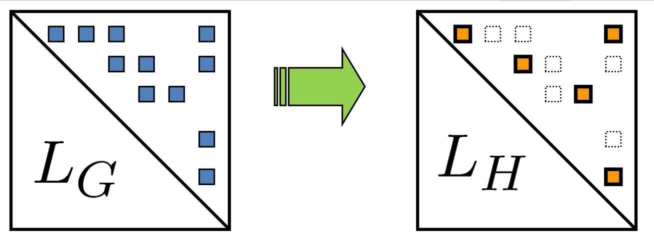
Spectral Graph Sparsification: Intuition

Let us consider unweighted graphs: $w_{ij} \in \{0, 1\}$

$$\mathbf{L}_G = \sum_{ij} w_{ij}\mathbf{L}_{ij} = \sum_{ij\in E}\mathbf{L}_{ij} = \sum_{ij\in E}(\boldsymbol{\delta}_i - \boldsymbol{\delta}_j)(\boldsymbol{\delta}_i - \boldsymbol{\delta}_j)^\mathsf{T} = \sum_{e\in E}\mathbf{b}_e\mathbf{b}_e^\mathsf{T}$$

We look for a **subgraph** H

$$\mathbf{L}_H = \sum_{e\in E} s_e\mathbf{b}_e\mathbf{b}_e^\mathsf{T} \quad \text{where } s_e \text{ is a new weight of edge e}$$



https://math.berkeley.edu/~nikhil/

Spectral Graph Sparsification: Intuition

We want $\quad (1-\varepsilon)\mathbf{L}_G \preceq \mathbf{L}_H \preceq (1+\varepsilon)\mathbf{L}_G$

Spectral Graph Sparsification: Intuition

$$\text{We want} \quad (1-\varepsilon)\mathbf{L}_G \preceq \mathbf{L}_H \preceq (1+\varepsilon)\mathbf{L}_G$$

Equivalent, given $\mathbf{L}_G = \sum_{e \in E} \mathbf{b}_e \mathbf{b}_e^\mathsf{T}$

Spectral Graph Sparsification: Intuition

$$\text{We want} \quad (1-\varepsilon)\mathbf{L}_G \preceq \mathbf{L}_H \preceq (1+\varepsilon)\mathbf{L}_G$$

Equivalent, given $\mathbf{L}_G = \sum_{e\in E} \mathbf{b}_e \mathbf{b}_e^\mathsf{T}$ find $\mathbf{s}$, s.t. $\mathbf{L}_G \preceq \sum_{e\in E} s_e \mathbf{b}_e \mathbf{b}_e^\mathsf{T} \preceq \kappa \cdot \mathbf{L}_G$

Spectral Graph Sparsification: Intuition

$$\text{We want} \quad (1-\varepsilon)\mathbf{L}_G \preceq \mathbf{L}_H \preceq (1+\varepsilon)\mathbf{L}_G$$

Equivalent, given $\mathbf{L}_G = \sum_{e \in E} \mathbf{b}_e \mathbf{b}_e^\mathsf{T}$ find $\mathbf{s}$, s.t. $\mathbf{L}_G \preceq \sum_{e \in E} s_e \mathbf{b}_e \mathbf{b}_e^\mathsf{T} \preceq \kappa \cdot \mathbf{L}_G$

Forget $\mathbf{L}$, given $\mathbf{A} = \sum_{e \in E} \mathbf{a}_e \mathbf{a}_e^\mathsf{T}$

Spectral Graph Sparsification: Intuition

$$\text{We want} \quad (1-\varepsilon)\mathbf{L}_G \preceq \mathbf{L}_H \preceq (1+\varepsilon)\mathbf{L}_G$$

Equivalent, given $\mathbf{L}_G = \sum_{e \in E} \mathbf{b}_e \mathbf{b}_e^\mathsf{T}$ find $\mathbf{s}$, s.t. $\mathbf{L}_G \preceq \sum_{e \in E} s_e \mathbf{b}_e \mathbf{b}_e^\mathsf{T} \preceq \kappa \cdot \mathbf{L}_G$

Forget $\mathbf{L}$, given $\mathbf{A} = \sum_{e \in E} \mathbf{a}_e \mathbf{a}_e^\mathsf{T}$ find $\mathbf{s}$, s.t. $\mathbf{A} \preceq \sum_{e \in E} s_e \mathbf{a}_e \mathbf{a}_e^\mathsf{T} \preceq \kappa \cdot \mathbf{A}$

Spectral Graph Sparsification: Intuition

We want $\quad (1-\varepsilon)\mathbf{L}_G \preceq \mathbf{L}_H \preceq (1+\varepsilon)\mathbf{L}_G$

Equivalent, given $\mathbf{L}_G = \sum_{e\in E} \mathbf{b}_e\mathbf{b}_e^\mathsf{T}$ find $\mathbf{s}$, s.t. $\mathbf{L}_G \preceq \sum_{e\in E} s_e\mathbf{b}_e\mathbf{b}_e^\mathsf{T} \preceq \kappa\cdot\mathbf{L}_G$

Forget $\mathbf{L}$, given $\mathbf{A} = \sum_{e\in E} \mathbf{a}_e\mathbf{a}_e^\mathsf{T}$ find $\mathbf{s}$, s.t. $\mathbf{A} \preceq \sum_{e\in E} s_e\mathbf{a}_e\mathbf{a}_e^\mathsf{T} \preceq \kappa \cdot \mathbf{A}$

Same as, given $\mathbf{I} = \sum_{e\in E} \mathbf{v}_e\mathbf{v}_e^\mathsf{T}$

Spectral Graph Sparsification: Intuition

We want $(1-\varepsilon)\mathbf{L}_G \preceq \mathbf{L}_H \preceq (1+\varepsilon)\mathbf{L}_G$

Equivalent, given $\mathbf{L}_G = \sum_{e \in E} \mathbf{b}_e \mathbf{b}_e^\mathsf{T}$ find $\mathbf{s}$, s.t. $\mathbf{L}_G \preceq \sum_{e \in E} s_e \mathbf{b}_e \mathbf{b}_e^\mathsf{T} \preceq \kappa \cdot \mathbf{L}_G$

Forget $\mathbf{L}$, given $\mathbf{A} = \sum_{e \in E} \mathbf{a}_e \mathbf{a}_e^\mathsf{T}$ find $\mathbf{s}$, s.t. $\mathbf{A} \preceq \sum_{e \in E} s_e \mathbf{a}_e \mathbf{a}_e^\mathsf{T} \preceq \kappa \cdot \mathbf{A}$

Same as, given $\mathbf{I} = \sum_{e \in E} \mathbf{v}_e \mathbf{v}_e^\mathsf{T}$ find $\mathbf{s}$, s.t. $\mathbf{I} \preceq \sum_{e \in E} s_e \mathbf{v}_e \mathbf{v}_e^\mathsf{T} \preceq \kappa \cdot \mathbf{I}$

Spectral Graph Sparsification: Intuition

We want $(1-\varepsilon)\mathbf{L}_G \preceq \mathbf{L}_H \preceq (1+\varepsilon)\mathbf{L}_G$

Equivalent, given $\mathbf{L}_G = \sum_{e\in E} \mathbf{b}_e\mathbf{b}_e^\mathsf{T}$ find $\mathbf{s}$, s.t. $\mathbf{L}_G \preceq \sum_{e\in E} s_e\mathbf{b}_e\mathbf{b}_e^\mathsf{T} \preceq \kappa\cdot\mathbf{L}_G$

Forget $\mathbf{L}$, given $\mathbf{A} = \sum_{e\in E} \mathbf{a}_e\mathbf{a}_e^\mathsf{T}$ find $\mathbf{s}$, s.t. $\mathbf{A} \preceq \sum_{e\in E} s_e\mathbf{a}_e\mathbf{a}_e^\mathsf{T} \preceq \kappa\cdot\mathbf{A}$

Same as, given $\mathbf{I} = \sum_{e\in E} \mathbf{v}_e\mathbf{v}_e^\mathsf{T}$ find $\mathbf{s}$, s.t. $\mathbf{I} \preceq \sum_{e\in E} s_e\mathbf{v}_e\mathbf{v}_e^\mathsf{T} \preceq \kappa\cdot\mathbf{I}$

How to get it?

Spectral Graph Sparsification: Intuition

We want $(1-\varepsilon)\mathbf{L}_G \preceq \mathbf{L}_H \preceq (1+\varepsilon)\mathbf{L}_G$

Equivalent, given $\mathbf{L}_G = \sum_{e\in E} \mathbf{b}_e\mathbf{b}_e^\mathsf{T}$ find $\mathbf{s}$, s.t. $\mathbf{L}_G \preceq \sum_{e\in E} s_e\mathbf{b}_e\mathbf{b}_e^\mathsf{T} \preceq \kappa\cdot\mathbf{L}_G$

Forget $\mathbf{L}$, given $\mathbf{A} = \sum_{e\in E} \mathbf{a}_e\mathbf{a}_e^\mathsf{T}$ find $\mathbf{s}$, s.t. $\mathbf{A} \preceq \sum_{e\in E} s_e\mathbf{a}_e\mathbf{a}_e^\mathsf{T} \preceq \kappa\cdot\mathbf{A}$

Same as, given $\mathbf{I} = \sum_{e\in E} \mathbf{v}_e\mathbf{v}_e^\mathsf{T}$ find $\mathbf{s}$, s.t. $\mathbf{I} \preceq \sum_{e\in E} s_e\mathbf{v}_e\mathbf{v}_e^\mathsf{T} \preceq \kappa\cdot\mathbf{I}$

How to get it? $\mathbf{v}_e \leftarrow \mathbf{A}^{-1/2}\mathbf{a}_e$

Spectral Graph Sparsification: Intuition

We want $\quad (1-\varepsilon)\mathbf{L}_G \preceq \mathbf{L}_H \preceq (1+\varepsilon)\mathbf{L}_G$

Equivalent, given $\mathbf{L}_G = \sum_{e\in E} \mathbf{b}_e\mathbf{b}_e^\mathsf{T}$ find $\mathbf{s}$, s.t. $\mathbf{L}_G \preceq \sum_{e\in E} s_e\mathbf{b}_e\mathbf{b}_e^\mathsf{T} \preceq \kappa\cdot\mathbf{L}_G$

Forget $\mathbf{L}$, given $\mathbf{A} = \sum_{e\in E} \mathbf{a}_e\mathbf{a}_e^\mathsf{T}$ find $\mathbf{s}$, s.t. $\mathbf{A} \preceq \sum_{e\in E} s_e\mathbf{a}_e\mathbf{a}_e^\mathsf{T} \preceq \kappa\cdot\mathbf{A}$

Same as, given $\mathbf{I} = \sum_{e\in E} \mathbf{v}_e\mathbf{v}_e^\mathsf{T}$ find $\mathbf{s}$, s.t. $\mathbf{I} \preceq \sum_{e\in E} s_e\mathbf{v}_e\mathbf{v}_e^\mathsf{T} \preceq \kappa\cdot\mathbf{I}$

How to get it? $\mathbf{v}_e \leftarrow \mathbf{A}^{-1/2}\mathbf{a}_e$

Then $\sum_{e\in E} s_e\mathbf{v}_e\mathbf{v}_e^\mathsf{T} \approx \mathbf{I} \iff \sum_{e\in E} s_e\mathbf{a}_e\mathbf{a}_e^\mathsf{T} \approx \mathbf{A}$

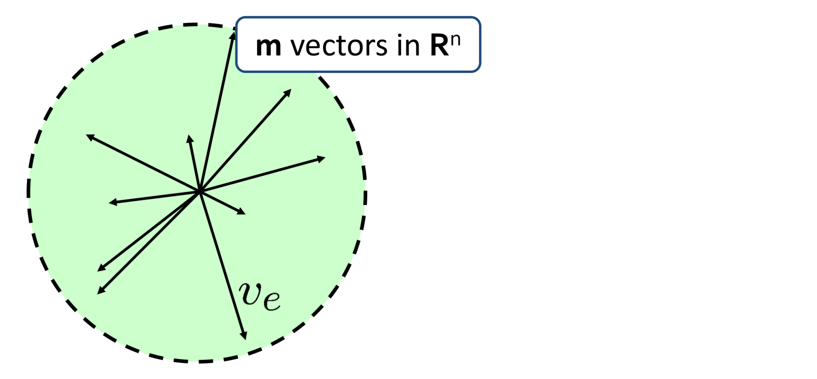
multiplying by $\mathbf{A}^{1/2}$ on both sides

Spectral Graph Sparsification: Intuition

How does $\sum_{e \in E} \mathbf{v}_e \mathbf{v}_e^\mathsf{T} = \mathbf{I}$ look like geometrically?

Spectral Graph Sparsification: Intuition

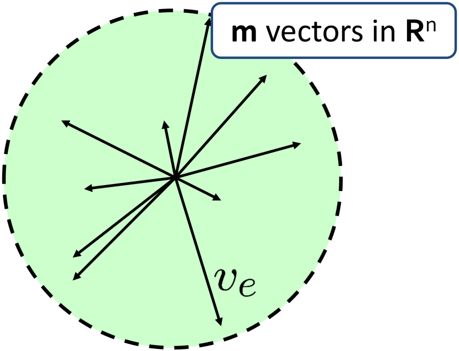
How does $\sum_{e \in E} \mathbf{v}_e \mathbf{v}_e^\mathsf{T} = \mathbf{I}$ look like geometrically?



Decomposition of identity: $\forall \mathbf{u}$ (unit vector): $\sum_{e \in E} (\mathbf{u}^\mathsf{T} \mathbf{v}_e)^2 = 1$

Spectral Graph Sparsification: Intuition

How does $\sum_{e \in E} \mathbf{v}_e \mathbf{v}_e^\mathsf{T} = \mathbf{I}$ look like geometrically?



Decomposition of identity: $\forall \mathbf{u}$ (unit vector): $\sum_{e \in E} (\mathbf{u}^\mathsf{T} \mathbf{v}_e)^2 = 1$

moment ellipse is a sphere

https://math.berkeley.edu/~nikhil/

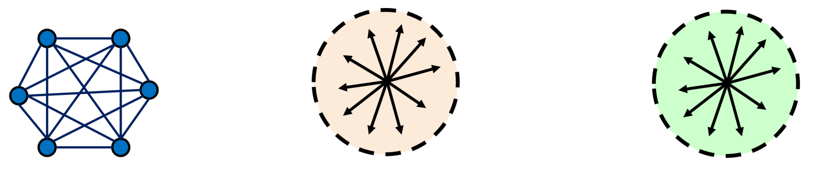
Spectral Graph Sparsification: Intuition

Example: What happens with K_n?

Spectral Graph Sparsification: Intuition

Example: What happens with K_n?

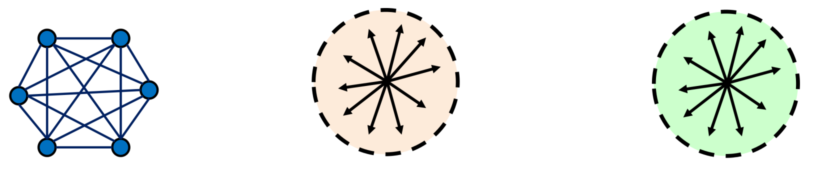
K_n graph | $\sum_{e \in E} \mathbf{b}_e \mathbf{b}_e^\mathsf{T} = \mathbf{L}_G$ | $\sum_{e \in E} \mathbf{v}_e \mathbf{v}_e^\mathsf{T} = \mathbf{I}$



Spectral Graph Sparsification: Intuition

Example: What happens with K_n?

K_n graph $\qquad \sum_{e \in E} \mathbf{b}_e \mathbf{b}_e^\mathsf{T} = \mathbf{L}_G \qquad \sum_{e \in E} \mathbf{v}_e \mathbf{v}_e^\mathsf{T} = \mathbf{I}$

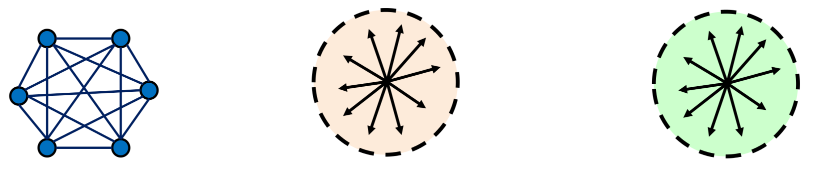


It is already isotropic! (looks like a sphere)

Spectral Graph Sparsification: Intuition

Example: What happens with K_n?

K_n graph $\qquad \sum_{e \in E} \mathbf{b}_e \mathbf{b}_e^\mathsf{T} = \mathbf{L}_G \qquad \sum_{e \in E} \mathbf{v}_e \mathbf{v}_e^\mathsf{T} = \mathbf{I}$



It is already isotropic! (looks like a sphere)

rescaling $\mathbf{v}_e = \mathbf{L}^{-1/2}\mathbf{b}_e$ does not change the shape

https://math.berkeley.edu/~nikhil/

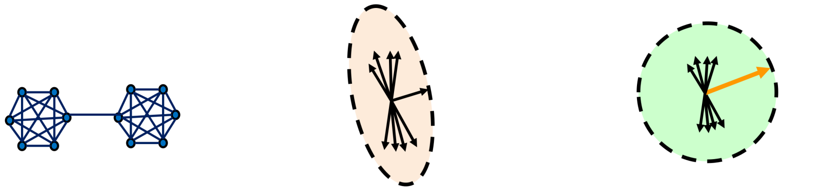
Spectral Graph Sparsification: Intuition

Example: What happens with a dumbbell?

Spectral Graph Sparsification: Intuition

Example: What happens with a dumbbell?

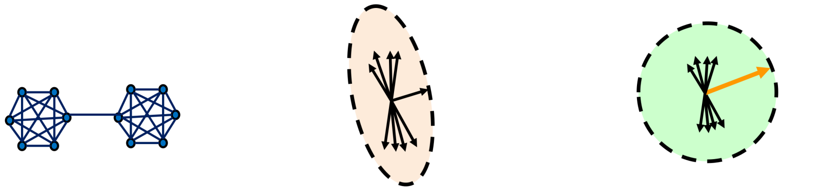
Dumbbell $\sum_{e \in E} \mathbf{b}_e \mathbf{b}_e^\mathsf{T} = \mathbf{L}_G$ $\sum_{e \in E} \mathbf{v}_e \mathbf{v}_e^\mathsf{T} = \mathbf{I}$



Spectral Graph Sparsification: Intuition

Example: What happens with a dumbbell?

Dumbbell $\qquad \sum_{e\in E} \mathbf{b}_e\mathbf{b}_e^{\mathsf{T}} = \mathbf{L}_G \qquad \sum_{e\in E} \mathbf{v}_e\mathbf{v}_e^{\mathsf{T}} = \mathbf{I}$

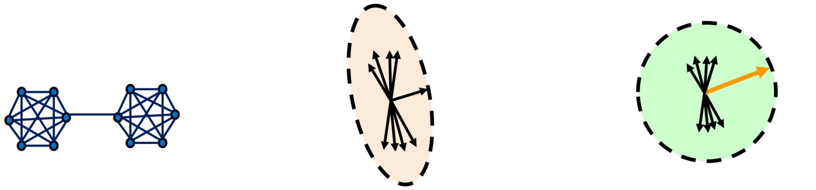


The vector corresponding to the link gets stretched!

Spectral Graph Sparsification: Intuition

Example: What happens with a dumbbell?

Dumbbell $\qquad \sum_{e \in E} \mathbf{b}_e \mathbf{b}_e^\mathsf{T} = \mathbf{L}_G \qquad \sum_{e \in E} \mathbf{v}_e \mathbf{v}_e^\mathsf{T} = \mathbf{I}$



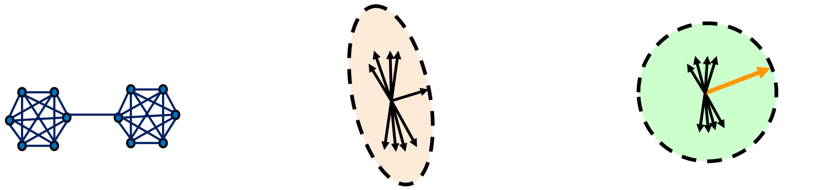
The vector corresponding to the link gets stretched!

because this transformation makes all the directions important

Spectral Graph Sparsification: Intuition

Example: What happens with a dumbbell?

Dumbbell $\sum_{e\in E} \mathbf{b}_e\mathbf{b}_e^{\mathsf{T}} = \mathbf{L}_G$ $\sum_{e\in E} \mathbf{v}_e\mathbf{v}_e^{\mathsf{T}} = \mathbf{I}$



The vector corresponding to the link gets stretched!

because this transformation makes all the directions important

rescaling reveals the vectors that are critical

https://math.berkeley.edu/~nikhil/

Michal Valko

michal.valko@inria.fr

Inria & ENS Paris-Saclay, MVA

https://misovalko.github.io/mva-ml-graphs.html