Graphs in Machine Learning

Cut Graph Sparsifiers

Benczur-Karger Algorithm

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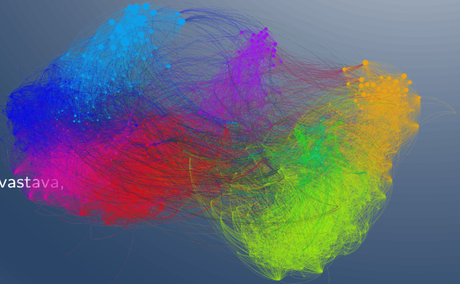
Inria & ENS Paris-Saclay, MVA

Partially based on material by: Rob Fergus, Nikhil Srivastava, Yiannis Koutis, Joshua Batson, Daniel Spielman



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Define G and H are $(1 \pm \varepsilon)$-**cut similar** when $\forall S$

$$(1-\varepsilon)\text{cut}_H(S) \leq \text{cut}_G(S) \leq (1+\varepsilon)\text{cut}_H(S)$$

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Is this always possible? Benczúr and Karger (1996): Yes!

$\forall \varepsilon \; \exists \; (1+\varepsilon)$-cut similar H with $\mathcal{O}(n \log n/\varepsilon^2)$ edges s.t. $E_H \subseteq E$ and computable in $\mathcal{O}(m \log^3 n + m \log n/\varepsilon^2)$ time n nodes, m edges

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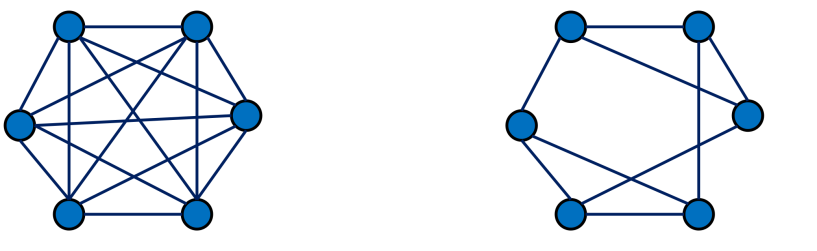
$G = K_n$ $\qquad$ $H =$ d-**regular** (random)



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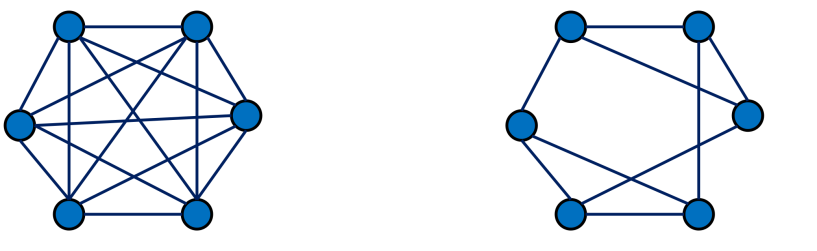

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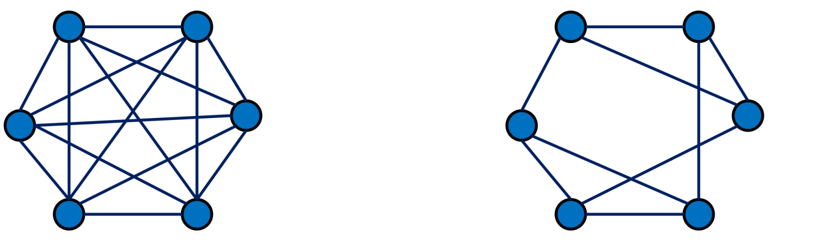

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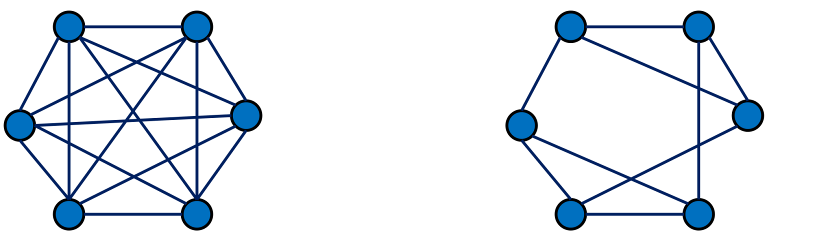

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$|E_G| = \mathcal{O}(n^2)$ $\qquad$ $|E_H| = \mathcal{O}(dn)$

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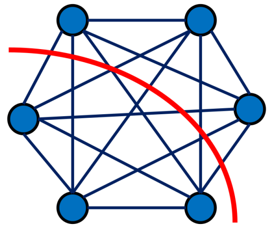

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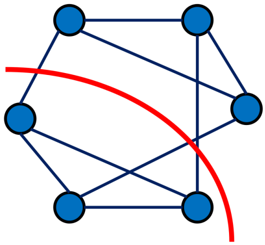

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Could be large :(What to do?

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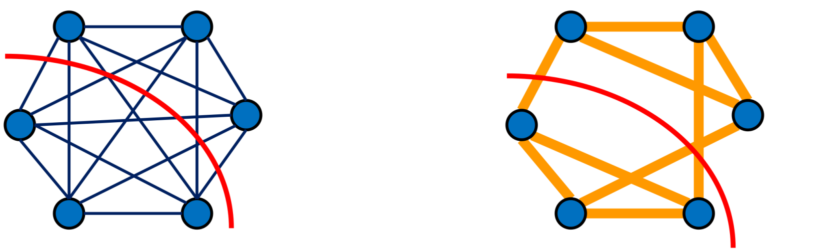

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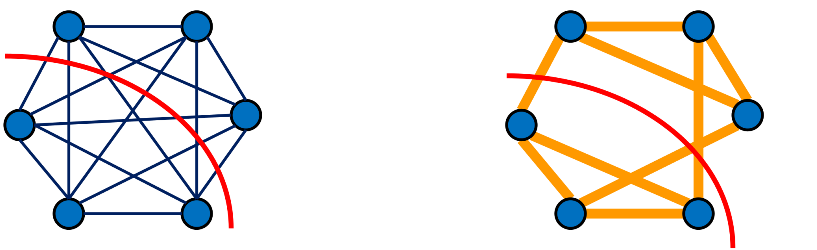

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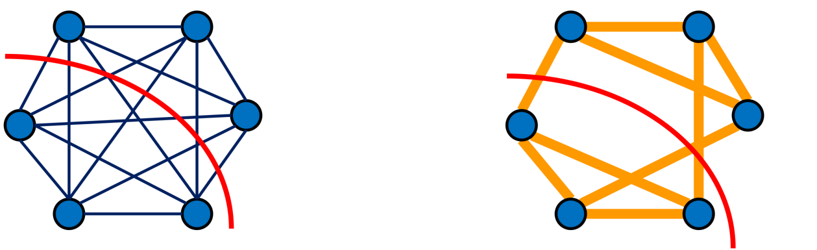

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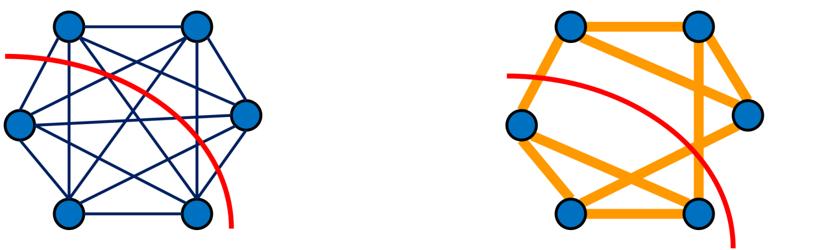

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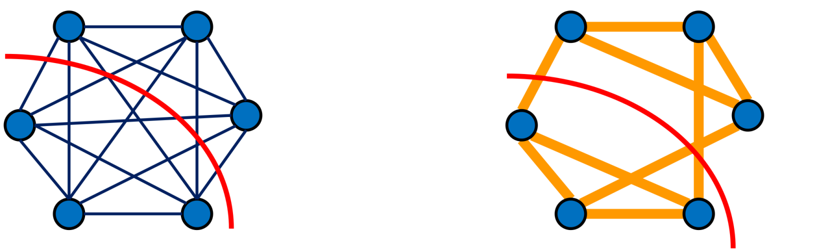

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Benczúr & Karger: Can find such H quickly for any G!

Graph Sparsification: What is **good** sparse?

Recall if $\mathbf{f} \in \{0,1\}^n$ represents S then $\mathbf{f}^\mathsf{T}\mathbf{L}_G\mathbf{f} =$

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Spectral sparsifiers are stronger!

but checking for spectral similarity is easier

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https://misovalko.github.io/mva-ml-graphs.html