Graphs in Machine Learning

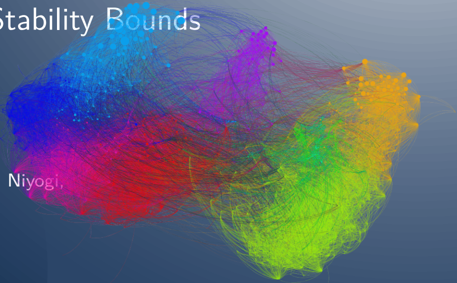
SSL Regularization and Stability

Regularized, Soft Harmonic, and Stability Bounds

Michal Valko

Inria & ENS Paris-Saclay, MVA

Partially based on material by: Mikhail Belkin, Partha Niyogi, Olivier Chapelle, Bernhard Schölkopf



SSL with Graphs: Regularized Harmonic Functions

$$f_i = p_i^{(+1)} - p_i^{(-1)}$$

SSL with Graphs: Regularized Harmonic Functions

$$f_i = p_i^{(+1)} - p_i^{(-1)} \qquad \Longrightarrow \quad f_i = \underbrace{|f_i|}_{\text{confidence}} \times \underbrace{\operatorname{sgn}(f_i)}_{\text{label}}$$

SSL with Graphs: Regularized Harmonic Functions

$$f_i = p_i^{(+1)} - p_i^{(-1)} \qquad\qquad \Longrightarrow \; f_i = \underbrace{|f_i|}_{\text{confidence}} \times \underbrace{\operatorname{sgn}(f_i)}_{\text{label}}$$

What if a nasty outlier sneaks in?

SSL with Graphs: Regularized Harmonic Functions

$$f_i = p_i^{(+1)} - p_i^{(-1)} \qquad \Longrightarrow \; f_i = \underbrace{|f_i|}_{\text{confidence}} \times \underbrace{\operatorname{sgn}(f_i)}_{\text{label}}$$

What if a nasty outlier sneaks in?

The prediction for the outlier can be hyperconfident :(

SSL with Graphs: Regularized Harmonic Functions

$$f_i = p_i^{(+1)} - p_i^{(-1)} \qquad \Longrightarrow \; f_i = \underbrace{|f_i|}_{\text{confidence}} \times \underbrace{\operatorname{sgn}(f_i)}_{\text{label}}$$

What if a nasty outlier sneaks in?

The prediction for the outlier can be hyperconfident :(

How to control the confidence of the inference?

SSL with Graphs: Regularized Harmonic Functions

$$f_i = p_i^{(+1)} - p_i^{(-1)} \qquad \Longrightarrow \quad f_i = \underbrace{|f_i|}_{\text{confidence}} \times \underbrace{\operatorname{sgn}(f_i)}_{\text{label}}$$

What if a nasty outlier sneaks in?

The prediction for the outlier can be hyperconfident :(

How to control the confidence of the inference?

Allow the random walk to **die**!

SSL with Graphs: Regularized Harmonic Functions

$$f_i = p_i^{(+1)} - p_i^{(-1)} \qquad \Longrightarrow \quad f_i = \underbrace{|f_i|}_{\text{confidence}} \times \underbrace{\mathrm{sgn}(f_i)}_{\text{label}}$$

What if a nasty outlier sneaks in?

The prediction for the outlier can be hyperconfident :(

How to control the confidence of the inference?

Allow the random walk to **die**!

We add a **sink** to the graph.

SSL with Graphs: Regularized Harmonic Functions

$$f_i = p_i^{(+1)} - p_i^{(-1)} \qquad \Longrightarrow \quad f_i = \underbrace{|f_i|}_{\text{confidence}} \times \underbrace{\mathrm{sgn}(f_i)}_{\text{label}}$$

What if a nasty outlier sneaks in?

The prediction for the outlier can be hyperconfident :(

How to control the confidence of the inference?

Allow the random walk to **die**!

We add a **sink** to the graph.

sink = artificial label node with value 0

SSL with Graphs: Regularized Harmonic Functions

$$f_i = p_i^{(+1)} - p_i^{(-1)} \qquad \Longrightarrow \quad f_i = \underbrace{|f_i|}_{\text{confidence}} \times \underbrace{\mathrm{sgn}(f_i)}_{\text{label}}$$

What if a nasty outlier sneaks in?

The prediction for the outlier can be hyperconfident :(

How to control the confidence of the inference?

Allow the random walk to **die**!

We add a **sink** to the graph.

sink = artificial label node with value 0

We connect it to every other vertex.

SSL with Graphs: Regularized Harmonic Functions

$$f_i = p_i^{(+1)} - p_i^{(-1)} \qquad \Longrightarrow \quad f_i = \underbrace{|f_i|}_{\text{confidence}} \times \underbrace{\operatorname{sgn}(f_i)}_{\text{label}}$$

What if a nasty outlier sneaks in?

The prediction for the outlier can be hyperconfident :(

How to control the confidence of the inference?

Allow the random walk to **die**!

We add a **sink** to the graph.

sink = artificial label node with value 0

We connect it to every other vertex.

SSL with Graphs: Regularized Harmonic Functions

$$f_i = p_i^{(+1)} - p_i^{(-1)} \qquad \Longrightarrow \; f_i = \underbrace{|f_i|}_{\text{confidence}} \times \underbrace{\operatorname{sgn}(f_i)}_{\text{label}}$$

What if a nasty outlier sneaks in?

The prediction for the outlier can be hyperconfident :(

How to control the confidence of the inference?

Allow the random walk to **die**!

We add a **sink** to the graph.

sink = artificial label node with value 0

We connect it to every other vertex.

SSL with Graphs: Regularized Harmonic Functions

How do we represent the sink in $\mathbf{L}$ explicitly?

SSL with Graphs: Regularized Harmonic Functions

How do we represent the sink in $\mathbf{L}$ explicitly?

Formally, to get the harmonic solution on the graph with sink ...

SSL with Graphs: Regularized Harmonic Functions

How do we represent the sink in $\mathbf{L}$ explicitly?

Formally, to get the harmonic solution on the graph with sink ...

$$\begin{bmatrix} \mathbf{L}_{ll} + \gamma_G \mathbf{I}_{n_l} & \mathbf{L}_{lu} & -\gamma_G \mathbf{1}_{n_l \times 1} \\ \mathbf{L}_{ul} & \mathbf{L}_{uu} + \gamma_G \mathbf{I}_{n_u} & -\gamma_G \mathbf{1}_{n_u \times 1} \\ -\gamma_G \mathbf{1}_{1 \times n_l} & -\gamma_G \mathbf{1}_{1 \times n_u} & n\gamma_G \end{bmatrix} \begin{bmatrix} \mathbf{f}_l \\ \mathbf{f}_u \\ 0 \end{bmatrix} = \begin{bmatrix} \dots \\ \mathbf{0}_u \\ \dots \end{bmatrix}$$

SSL with Graphs: Regularized Harmonic Functions

How do we represent the sink in $\mathbf{L}$ explicitly?

Formally, to get the harmonic solution on the graph with sink ...

$$\begin{bmatrix} \mathbf{L}_{ll} + \gamma_G \mathbf{I}_{n_l} & \mathbf{L}_{lu} & -\gamma_G \mathbf{1}_{n_l \times 1} \\ \mathbf{L}_{ul} & \mathbf{L}_{uu} + \gamma_G \mathbf{I}_{n_u} & -\gamma_G \mathbf{1}_{n_u \times 1} \\ -\gamma_G \mathbf{1}_{1 \times n_l} & -\gamma_G \mathbf{1}_{1 \times n_u} & n\gamma_G \end{bmatrix} \begin{bmatrix} \mathbf{f}_l \\ \mathbf{f}_u \\ 0 \end{bmatrix} = \begin{bmatrix} \dots \\ \mathbf{0}_u \\ \dots \end{bmatrix}$$

$$\mathbf{L}_{ul}\mathbf{f}_l + \left(\mathbf{L}_{uu} + \gamma_G \mathbf{I}_{n_u}\right)\mathbf{f}_u = \mathbf{0}_u$$

SSL with Graphs: Regularized Harmonic Functions

How do we represent the sink in $\mathbf{L}$ explicitly?

Formally, to get the harmonic solution on the graph with sink ...

$$\begin{bmatrix} \mathbf{L}_{ll} + \gamma_G \mathbf{I}_{n_l} & \mathbf{L}_{lu} & -\gamma_G \mathbf{1}_{n_l \times 1} \\ \mathbf{L}_{ul} & \mathbf{L}_{uu} + \gamma_G \mathbf{I}_{n_u} & -\gamma_G \mathbf{1}_{n_u \times 1} \\ -\gamma_G \mathbf{1}_{1 \times n_l} & -\gamma_G \mathbf{1}_{1 \times n_u} & n\gamma_G \end{bmatrix} \begin{bmatrix} \mathbf{f}_l \\ \mathbf{f}_u \\ 0 \end{bmatrix} = \begin{bmatrix} \dots \\ \mathbf{0}_u \\ \dots \end{bmatrix}$$

$$\mathbf{L}_{ul}\mathbf{f}_l + \left(\mathbf{L}_{uu} + \gamma_G \mathbf{I}_{n_u}\right)\mathbf{f}_u = \mathbf{0}_u$$

...which is the same if we disregard the last column and row ...

$$\begin{bmatrix} \mathbf{L}_{ll} + \gamma_G \mathbf{I}_{n_l} & \mathbf{L}_{lu} \\ \mathbf{L}_{ul} & \mathbf{L}_{uu} + \gamma_G \mathbf{I}_{n_u} \end{bmatrix} \begin{bmatrix} \mathbf{f}_l \\ \mathbf{f}_u \end{bmatrix} = \begin{bmatrix} \dots \\ \mathbf{0}_u \end{bmatrix}$$

SSL with Graphs: Regularized Harmonic Functions

How do we represent the sink in $\mathbf{L}$ explicitly?

Formally, to get the harmonic solution on the graph with sink ...

$$\begin{bmatrix} \mathbf{L}_{ll}+\gamma_G\mathbf{I}_{n_l} & \mathbf{L}_{lu} & -\gamma_G\mathbf{1}_{n_l\times 1} \\ \mathbf{L}_{ul} & \mathbf{L}_{uu}+\gamma_G\mathbf{I}_{n_u} & -\gamma_G\mathbf{1}_{n_u\times 1} \\ -\gamma_G\mathbf{1}_{1\times n_l} & -\gamma_G\mathbf{1}_{1\times n_u} & n\gamma_G \end{bmatrix} \begin{bmatrix} \mathbf{f}_l \\ \mathbf{f}_u \\ 0 \end{bmatrix} = \begin{bmatrix} \dots \\ \mathbf{0}_u \\ \dots \end{bmatrix}$$

$$\mathbf{L}_{ul}\mathbf{f}_l + \left(\mathbf{L}_{uu}+\gamma_G\mathbf{I}_{n_u}\right)\mathbf{f}_u = \mathbf{0}_u$$

...which is the same if we disregard the last column and row ...

$$\begin{bmatrix} \mathbf{L}_{ll}+\gamma_G\mathbf{I}_{n_l} & \mathbf{L}_{lu} \\ \mathbf{L}_{ul} & \mathbf{L}_{uu}+\gamma_G\mathbf{I}_{n_u} \end{bmatrix} \begin{bmatrix} \mathbf{f}_l \\ \mathbf{f}_u \end{bmatrix} = \begin{bmatrix} \dots \\ \mathbf{0}_u \end{bmatrix}$$

...and therefore we simply add γ_G to the diagonal of $\mathbf{L}$!

SSL with Graphs: Regularized Harmonic Functions

How do we compute this **regularized** random walk?

SSL with Graphs: Regularized Harmonic Functions

How do we compute this **regularized** random walk?

$$\mathbf{f}_u = \left(\mathbf{L}_{uu} + \gamma_g \mathbf{I}\right)^{-1} \left(\mathbf{W}_{ul}\mathbf{f}_l\right)$$

SSL with Graphs: Regularized Harmonic Functions

How do we compute this **regularized** random walk?

$$\mathbf{f}_u = \left(\mathbf{L}_{uu} + \gamma_g \mathbf{I}\right)^{-1} \left(\mathbf{W}_{ul}\mathbf{f}_l\right)$$

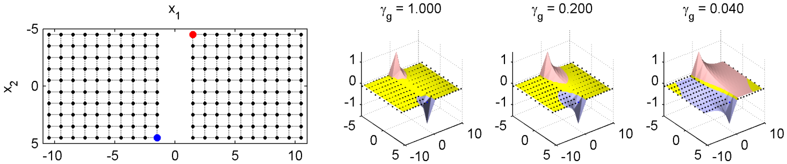
How does γ_g influence HS?

SSL with Graphs: Regularized Harmonic Functions

How do we compute this **regularized** random walk?

$$\mathbf{f}_u = \left(\mathbf{L}_{uu} + \gamma_g \mathbf{I}\right)^{-1} \left(\mathbf{W}_{ul}\mathbf{f}_l\right)$$

How does γ_g influence HS?

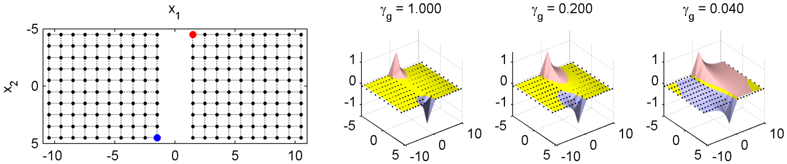


SSL with Graphs: Regularized Harmonic Functions

How do we compute this **regularized** random walk?

$$\mathbf{f}_u = (\mathbf{L}_{uu} + \gamma_g \mathbf{I})^{-1} (\mathbf{W}_{ul}\mathbf{f}_l)$$

How does γ_g influence HS?



What happens to sneaky outliers?

SSL with Graphs: Soft Harmonic Functions

Regularized HS objective with $\mathbf{Q} = \mathbf{L} + \gamma_g \mathbf{I}$:

SSL with Graphs: Soft Harmonic Functions

Regularized HS objective with $\mathbf{Q} = \mathbf{L} + \gamma_g \mathbf{I}$:

$$\min_{\mathbf{f} \in \mathbb{R}^{n_l + n_u}} \infty \sum_{i=1}^{n_l} \left(f(\mathbf{x}_i) - y_i \right)^2 + \lambda \mathbf{f}^\mathsf{T} \mathbf{Q} \mathbf{f}$$

SSL with Graphs: Soft Harmonic Functions

Regularized HS objective with $\mathbf{Q} = \mathbf{L} + \gamma_g \mathbf{I}$:

$$\min_{\mathbf{f} \in \mathbb{R}^{n_l+n_u}} \infty \sum_{i=1}^{n_l} \left(f(\mathbf{x}_i) - y_i\right)^2 + \lambda \mathbf{f}^\mathsf{T} \mathbf{Q} \mathbf{f}$$

What if we do not really believe that $f(\mathbf{x}_i) = y_i$, $\forall i$?

SSL with Graphs: Soft Harmonic Functions

Regularized HS objective with $\mathbf{Q} = \mathbf{L} + \gamma_g \mathbf{I}$:

$$\min_{\mathbf{f} \in \mathbb{R}^{n_l + n_u}} \infty \sum_{i=1}^{n_l} (f(\mathbf{x}_i) - y_i)^2 + \lambda \mathbf{f}^\mathsf{T} \mathbf{Q} \mathbf{f}$$

What if we do not really believe that $f(\mathbf{x}_i) = y_i$, $\forall i$?

$$\mathbf{f}^\star = \min_{\mathbf{f} \in \mathbb{R}^N} (\mathbf{f} - \mathbf{y})^\mathsf{T} \mathbf{C} (\mathbf{f} - \mathbf{y}) + \mathbf{f}^\mathsf{T} \mathbf{Q} \mathbf{f}$$

SSL with Graphs: Soft Harmonic Functions

Regularized HS objective with $\mathbf{Q} = \mathbf{L} + \gamma_g \mathbf{I}$:

$$\min_{\mathbf{f} \in \mathbb{R}^{n_l + n_u}} \infty \sum_{i=1}^{n_l} \left(f(\mathbf{x}_i) - y_i\right)^2 + \lambda \mathbf{f}^\mathsf{T} \mathbf{Q} \mathbf{f}$$

What if we do not really believe that $f(\mathbf{x}_i) = y_i$, $\forall i$?

$$\mathbf{f}^\star = \min_{\mathbf{f} \in \mathbb{R}^N} (\mathbf{f} - \mathbf{y})^\mathsf{T} \mathbf{C} (\mathbf{f} - \mathbf{y}) + \mathbf{f}^\mathsf{T} \mathbf{Q} \mathbf{f}$$

$\mathbf{C}$ is diagonal with $C_{ii} = \begin{cases} c_l & \text{for labeled examples} \\ c_u & \text{otherwise.} \end{cases}$

SSL with Graphs: Soft Harmonic Functions

Regularized HS objective with $\mathbf{Q} = \mathbf{L} + \gamma_g \mathbf{I}$:

$$\min_{\mathbf{f} \in \mathbb{R}^{n_l+n_u}} \infty \sum_{i=1}^{n_l} \left(f(\mathbf{x}_i) - y_i\right)^2 + \lambda \mathbf{f}^\top \mathbf{Q} \mathbf{f}$$

What if we do not really believe that $f(\mathbf{x}_i) = y_i$, $\forall i$?

$$\mathbf{f}^\star = \min_{\mathbf{f} \in \mathbb{R}^N} (\mathbf{f} - \mathbf{y})^\top \mathbf{C} (\mathbf{f} - \mathbf{y}) + \mathbf{f}^\top \mathbf{Q} \mathbf{f}$$

$\mathbf{C}$ is diagonal with $C_{ii} = \begin{cases} c_l & \text{for labeled examples} \\ c_u & \text{otherwise.} \end{cases}$

$\mathbf{y} \equiv$ pseudo-targets with $y_i = \begin{cases} \text{true label} & \text{for labeled examples} \\ 0 & \text{otherwise.} \end{cases}$

SSL with Graphs: Soft Harmonic Functions

$$\mathbf{f}^\star = \min_{\mathbf{f} \in \mathbb{R}^n} (\mathbf{f} - \mathbf{y})^\mathsf{T} \mathbf{C} (\mathbf{f} - \mathbf{y}) + \mathbf{f}^\mathsf{T} \mathbf{Q} \mathbf{f}$$

Closed form **soft harmonic solution**:

SSL with Graphs: Soft Harmonic Functions

$$\mathbf{f}^\star = \min_{\mathbf{f}\in\mathbb{R}^n} (\mathbf{f}-\mathbf{y})^\mathsf{T}\mathbf{C}(\mathbf{f}-\mathbf{y}) + \mathbf{f}^\mathsf{T}\mathbf{Q}\mathbf{f}$$

Closed form **soft harmonic solution**:

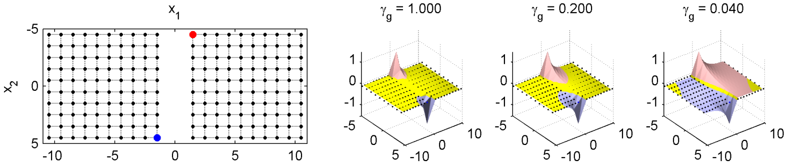
$$\mathbf{f}^\star = (\mathbf{C}^{-1}\mathbf{Q}+\mathbf{I})^{-1}\mathbf{y}$$

SSL with Graphs: Soft Harmonic Functions

$$\mathbf{f}^\star = \min_{\mathbf{f}\in\mathbb{R}^n} (\mathbf{f}-\mathbf{y})^\mathsf{T}\mathbf{C}(\mathbf{f}-\mathbf{y}) + \mathbf{f}^\mathsf{T}\mathbf{Q}\mathbf{f}$$

Closed form **soft harmonic solution**:

$$\mathbf{f}^\star = (\mathbf{C}^{-1}\mathbf{Q} + \mathbf{I})^{-1}\mathbf{y}$$

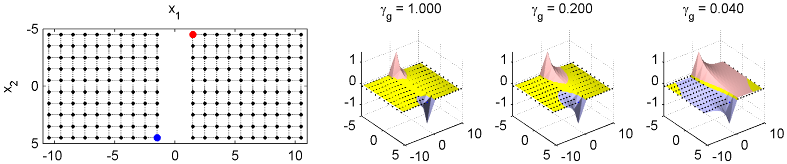


SSL with Graphs: Soft Harmonic Functions

$$\mathbf{f}^\star = \min_{\mathbf{f}\in\mathbb{R}^n} (\mathbf{f}-\mathbf{y})^\mathsf{T}\mathbf{C}(\mathbf{f}-\mathbf{y}) + \mathbf{f}^\mathsf{T}\mathbf{Q}\mathbf{f}$$

Closed form **soft harmonic solution**:

$$\mathbf{f}^\star = (\mathbf{C}^{-1}\mathbf{Q}+\mathbf{I})^{-1}\mathbf{y}$$



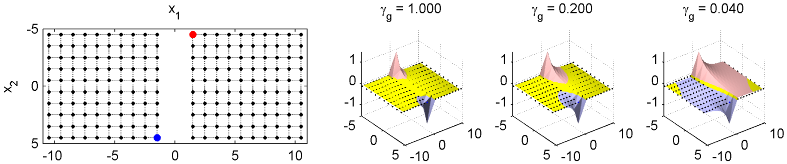
What are the differences between hard and soft?

SSL with Graphs: Soft Harmonic Functions

$$\mathbf{f}^\star = \min_{\mathbf{f} \in \mathbb{R}^n} (\mathbf{f} - \mathbf{y})^\mathsf{T} \mathbf{C} (\mathbf{f} - \mathbf{y}) + \mathbf{f}^\mathsf{T} \mathbf{Q} \mathbf{f}$$

Closed form **soft harmonic solution**:

$$\mathbf{f}^\star = (\mathbf{C}^{-1}\mathbf{Q} + \mathbf{I})^{-1}\mathbf{y}$$



What are the differences between hard and soft?

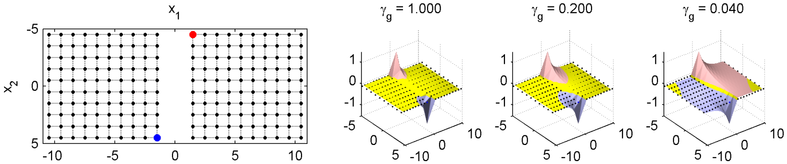
Not much different in practice.

SSL with Graphs: Soft Harmonic Functions

$$\mathbf{f}^\star = \min_{\mathbf{f}\in\mathbb{R}^n} (\mathbf{f}-\mathbf{y})^\mathsf{T}\mathbf{C}(\mathbf{f}-\mathbf{y}) + \mathbf{f}^\mathsf{T}\mathbf{Q}\mathbf{f}$$

Closed form **soft harmonic solution**:

$$\mathbf{f}^\star = (\mathbf{C}^{-1}\mathbf{Q}+\mathbf{I})^{-1}\mathbf{y}$$



What are the differences between hard and soft?

Not much different in practice.

Provable generalization guarantees for the soft one.

SSL with Graphs: Stability Bounds

$$\mathbf{f}^{\star} = \min_{\mathbf{f} \in \mathbb{R}^N} (\mathbf{f} - \mathbf{y})^{\mathsf{T}} \mathbf{C} (\mathbf{f} - \mathbf{y}) + \mathbf{f}^{\mathsf{T}} \mathbf{Q} \mathbf{f}$$

SSL with Graphs: Stability Bounds

$$\mathbf{f}^\star = \min_{\mathbf{f}\in\mathbb{R}^N} (\mathbf{f}-\mathbf{y})^\mathsf{T}\mathbf{C}(\mathbf{f}-\mathbf{y}) + \mathbf{f}^\mathsf{T}\mathbf{Q}\mathbf{f}$$

Think about **stability** of this solution.

SSL with Graphs: Stability Bounds

$$\mathbf{f}^\star = \min_{\mathbf{f}\in\mathbb{R}^N} (\mathbf{f}-\mathbf{y})^\mathsf{T}\mathbf{C}(\mathbf{f}-\mathbf{y}) + \mathbf{f}^\mathsf{T}\mathbf{Q}\mathbf{f}$$

Think about **stability** of this solution.

Consider two datasets differing in exactly one *labeled* point.

SSL with Graphs: Stability Bounds

$$\mathbf{f}^\star = \min_{\mathbf{f} \in \mathbb{R}^N} (\mathbf{f} - \mathbf{y})^\mathsf{T} \mathbf{C} (\mathbf{f} - \mathbf{y}) + \mathbf{f}^\mathsf{T} \mathbf{Q} \mathbf{f}$$

Think about **stability** of this solution.

Consider two datasets differing in exactly one *labeled* point.

$\mathcal{C}_1 = \mathbf{C}_1^{-1}\mathbf{Q} + \mathbf{I}$ and $\mathcal{C}_2 = \mathbf{C}_2^{-1}\mathbf{Q} + \mathbf{I}$

SSL with Graphs: Stability Bounds

$$\mathbf{f}^\star = \min_{\mathbf{f}\in\mathbb{R}^N} (\mathbf{f}-\mathbf{y})^\mathsf{T}\mathbf{C}(\mathbf{f}-\mathbf{y}) + \mathbf{f}^\mathsf{T}\mathbf{Q}\mathbf{f}$$

Think about **stability** of this solution.

Consider two datasets differing in exactly one *labeled* point.

$\mathcal{C}_1 = \mathbf{C}_1^{-1}\mathbf{Q} + \mathbf{I}$ and $\mathcal{C}_2 = \mathbf{C}_2^{-1}\mathbf{Q} + \mathbf{I}$

What is the maximal difference in the solutions?

SSL with Graphs: Stability Bounds

$$\mathbf{f}^{\star} = \min_{\mathbf{f} \in \mathbb{R}^{N}} (\mathbf{f} - \mathbf{y})^{\mathsf{T}} \mathbf{C} (\mathbf{f} - \mathbf{y}) + \mathbf{f}^{\mathsf{T}} \mathbf{Q} \mathbf{f}$$

Think about **stability** of this solution.

Consider two datasets differing in exactly one *labeled* point.

$\mathcal{C}_1 = \mathbf{C}_1^{-1}\mathbf{Q} + \mathbf{I}$ and $\mathcal{C}_2 = \mathbf{C}_2^{-1}\mathbf{Q} + \mathbf{I}$

What is the maximal difference in the solutions?

$$\mathbf{f}_2^{\star} - \mathbf{f}_1^{\star}$$

SSL with Graphs: Stability Bounds

$$\mathbf{f}^\star = \min_{\mathbf{f}\in\mathbb{R}^N} (\mathbf{f}-\mathbf{y})^\mathsf{T}\mathbf{C}(\mathbf{f}-\mathbf{y}) + \mathbf{f}^\mathsf{T}\mathbf{Q}\mathbf{f}$$

Think about **stability** of this solution.

Consider two datasets differing in exactly one *labeled* point.

$\mathcal{C}_1 = \mathbf{C}_1^{-1}\mathbf{Q} + \mathbf{I}$ and $\mathcal{C}_2 = \mathbf{C}_2^{-1}\mathbf{Q} + \mathbf{I}$

What is the maximal difference in the solutions?

$$\mathbf{f}_2^\star - \mathbf{f}_1^\star = \mathcal{C}_2^{-1}\mathbf{y}_2 - \mathcal{C}_1^{-1}\mathbf{y}_1$$

SSL with Graphs: Stability Bounds

$$\mathbf{f}^\star = \min_{\mathbf{f}\in\mathbb{R}^N} (\mathbf{f}-\mathbf{y})^\mathsf{T}\mathbf{C}(\mathbf{f}-\mathbf{y}) + \mathbf{f}^\mathsf{T}\mathbf{Q}\mathbf{f}$$

Think about **stability** of this solution.
Consider two datasets differing in exactly one *labeled* point.
$\mathcal{C}_1 = \mathbf{C}_1^{-1}\mathbf{Q} + \mathbf{I}$ and $\mathcal{C}_2 = \mathbf{C}_2^{-1}\mathbf{Q} + \mathbf{I}$

What is the maximal difference in the solutions?

$$\begin{aligned}\mathbf{f}_2^\star - \mathbf{f}_1^\star &= \mathcal{C}_2^{-1}\mathbf{y}_2 - \mathcal{C}_1^{-1}\mathbf{y}_1 \\ &= \mathcal{C}_2^{-1}(\mathbf{y}_2 - \mathbf{y}_1) - \left(\mathcal{C}_1^{-1} - \mathcal{C}_2^{-1}\right)\mathbf{y}_1\end{aligned}$$

SSL with Graphs: Stability Bounds

$$\mathbf{f}^\star = \min_{\mathbf{f} \in \mathbb{R}^N} (\mathbf{f} - \mathbf{y})^\mathsf{T} \mathbf{C} (\mathbf{f} - \mathbf{y}) + \mathbf{f}^\mathsf{T} \mathbf{Q} \mathbf{f}$$

Think about **stability** of this solution.
Consider two datasets differing in exactly one *labeled* point.
$\mathcal{C}_1 = \mathbf{C}_1^{-1}\mathbf{Q} + \mathbf{I}$ and $\mathcal{C}_2 = \mathbf{C}_2^{-1}\mathbf{Q} + \mathbf{I}$

What is the maximal difference in the solutions?

$$\begin{aligned}
\mathbf{f}_2^\star - \mathbf{f}_1^\star &= \mathcal{C}_2^{-1}\mathbf{y}_2 - \mathcal{C}_1^{-1}\mathbf{y}_1 \\
&= \mathcal{C}_2^{-1}(\mathbf{y}_2 - \mathbf{y}_1) - \left(\mathcal{C}_1^{-1} - \mathcal{C}_2^{-1}\right)\mathbf{y}_1 \\
&= \mathcal{C}_2^{-1}(\mathbf{y}_2 - \mathbf{y}_1) - \left(\mathcal{C}_1^{-1}\left[\left(\mathbf{C}_1^{-1} - \mathbf{C}_2^{-1}\right)\mathbf{Q}\right]\mathcal{C}_2^{-1}\right)\mathbf{y}_1
\end{aligned}$$

SSL with Graphs: Stability Bounds

$$\mathbf{f}^\star = \min_{\mathbf{f}\in\mathbb{R}^N} (\mathbf{f}-\mathbf{y})^\mathsf{T}\mathbf{C}(\mathbf{f}-\mathbf{y}) + \mathbf{f}^\mathsf{T}\mathbf{Q}\mathbf{f}$$

Think about **stability** of this solution.
Consider two datasets differing in exactly one *labeled* point.
$\mathcal{C}_1 = \mathbf{C}_1^{-1}\mathbf{Q} + \mathbf{I}$ and $\mathcal{C}_2 = \mathbf{C}_2^{-1}\mathbf{Q} + \mathbf{I}$

What is the maximal difference in the solutions?

$$\begin{aligned}
\mathbf{f}_2^\star - \mathbf{f}_1^\star &= \mathcal{C}_2^{-1}\mathbf{y}_2 - \mathcal{C}_1^{-1}\mathbf{y}_1 \\
&= \mathcal{C}_2^{-1}(\mathbf{y}_2 - \mathbf{y}_1) - \left(\mathcal{C}_1^{-1} - \mathcal{C}_2^{-1}\right)\mathbf{y}_1 \\
&= \mathcal{C}_2^{-1}(\mathbf{y}_2 - \mathbf{y}_1) - \left(\mathcal{C}_1^{-1}\left[\left(\mathbf{C}_1^{-1} - \mathbf{C}_2^{-1}\right)\mathbf{Q}\right]\mathcal{C}_2^{-1}\right)\mathbf{y}_1
\end{aligned}$$

SSL with Graphs: Stability Bounds

$$\mathbf{f}^\star = \min_{\mathbf{f}\in\mathbb{R}^N} (\mathbf{f}-\mathbf{y})^\mathsf{T}\mathbf{C}(\mathbf{f}-\mathbf{y}) + \mathbf{f}^\mathsf{T}\mathbf{Q}\mathbf{f}$$

Think about **stability** of this solution.

Consider two datasets differing in exactly one *labeled* point.

$\mathcal{C}_1 = \mathbf{C}_1^{-1}\mathbf{Q} + \mathbf{I}$ and $\mathcal{C}_2 = \mathbf{C}_2^{-1}\mathbf{Q} + \mathbf{I}$

What is the maximal difference in the solutions?

$$\begin{aligned}
\mathbf{f}_2^\star - \mathbf{f}_1^\star &= \mathcal{C}_2^{-1}\mathbf{y}_2 - \mathcal{C}_1^{-1}\mathbf{y}_1 \\
&= \mathcal{C}_2^{-1}(\mathbf{y}_2 - \mathbf{y}_1) - \left(\mathcal{C}_1^{-1} - \mathcal{C}_2^{-1}\right)\mathbf{y}_1 \\
&= \mathcal{C}_2^{-1}(\mathbf{y}_2 - \mathbf{y}_1) - \left(\mathcal{C}_1^{-1}\left[\left(\mathbf{C}_1^{-1} - \mathbf{C}_2^{-1}\right)\mathbf{Q}\right]\mathcal{C}_2^{-1}\right)\mathbf{y}_1
\end{aligned}$$

Note that $\mathbf{v} \in \mathbb{R}^{N\times 1}$, $\lambda_m(A)\|\mathbf{v}\|_2 \le \|A\mathbf{v}\|_2 \le \lambda_M(A)\|\mathbf{v}\|_2$

SSL with Graphs: Stability Bounds

$$\mathbf{f}^\star = \min_{\mathbf{f}\in\mathbb{R}^N} (\mathbf{f}-\mathbf{y})^\mathsf{T}\mathbf{C}(\mathbf{f}-\mathbf{y}) + \mathbf{f}^\mathsf{T}\mathbf{Q}\mathbf{f}$$

Think about **stability** of this solution.
Consider two datasets differing in exactly one *labeled* point.
$\mathcal{C}_1 = \mathbf{C}_1^{-1}\mathbf{Q} + \mathbf{I}$ and $\mathcal{C}_2 = \mathbf{C}_2^{-1}\mathbf{Q} + \mathbf{I}$

What is the maximal difference in the solutions?

$$\begin{aligned}
\mathbf{f}_2^\star - \mathbf{f}_1^\star &= \mathcal{C}_2^{-1}\mathbf{y}_2 - \mathcal{C}_1^{-1}\mathbf{y}_1 \\
&= \mathcal{C}_2^{-1}(\mathbf{y}_2 - \mathbf{y}_1) - \left(\mathcal{C}_1^{-1} - \mathcal{C}_2^{-1}\right)\mathbf{y}_1 \\
&= \mathcal{C}_2^{-1}(\mathbf{y}_2 - \mathbf{y}_1) - \left(\mathcal{C}_1^{-1}\left[\left(\mathbf{C}_1^{-1} - \mathbf{C}_2^{-1}\right)\mathbf{Q}\right]\mathcal{C}_2^{-1}\right)\mathbf{y}_1
\end{aligned}$$

Note that $\mathbf{v} \in \mathbb{R}^{N\times 1}$, $\lambda_m(A)\|\mathbf{v}\|_2 \le \|A\mathbf{v}\|_2 \le \lambda_M(A)\|\mathbf{v}\|_2$

$$\|\mathbf{f}_2^\star - \mathbf{f}_1^\star\|_2$$

SSL with Graphs: Stability Bounds

$$\mathbf{f}^\star = \min_{\mathbf{f}\in\mathbb{R}^N} (\mathbf{f}-\mathbf{y})^\mathsf{T}\mathbf{C}(\mathbf{f}-\mathbf{y}) + \mathbf{f}^\mathsf{T}\mathbf{Q}\mathbf{f}$$

Think about **stability** of this solution.
Consider two datasets differing in exactly one *labeled* point.
$\mathcal{C}_1 = \mathbf{C}_1^{-1}\mathbf{Q} + \mathbf{I}$ and $\mathcal{C}_2 = \mathbf{C}_2^{-1}\mathbf{Q} + \mathbf{I}$

What is the maximal difference in the solutions?

$$\begin{aligned}
\mathbf{f}_2^\star - \mathbf{f}_1^\star &= \mathcal{C}_2^{-1}\mathbf{y}_2 - \mathcal{C}_1^{-1}\mathbf{y}_1 \\
&= \mathcal{C}_2^{-1}(\mathbf{y}_2 - \mathbf{y}_1) - \left(\mathcal{C}_1^{-1} - \mathcal{C}_2^{-1}\right)\mathbf{y}_1 \\
&= \mathcal{C}_2^{-1}(\mathbf{y}_2 - \mathbf{y}_1) - \left(\mathcal{C}_1^{-1}\left[\left(\mathbf{C}_1^{-1} - \mathbf{C}_2^{-1}\right)\mathbf{Q}\right]\mathcal{C}_2^{-1}\right)\mathbf{y}_1
\end{aligned}$$

Note that $\mathbf{v} \in \mathbb{R}^{N\times 1}$, $\lambda_m(A)\|\mathbf{v}\|_2 \le \|A\mathbf{v}\|_2 \le \lambda_M(A)\|\mathbf{v}\|_2$

$$\|\mathbf{f}_2^\star - \mathbf{f}_1^\star\|_2 \le \frac{\|\mathbf{y}_2 - \mathbf{y}_1\|_2}{\lambda_m(\mathcal{C}_2)}$$

SSL with Graphs: Stability Bounds

$$\mathbf{f}^\star = \min_{\mathbf{f}\in\mathbb{R}^N} (\mathbf{f}-\mathbf{y})^\mathsf{T}\mathbf{C}(\mathbf{f}-\mathbf{y}) + \mathbf{f}^\mathsf{T}\mathbf{Q}\mathbf{f}$$

Think about **stability** of this solution.
Consider two datasets differing in exactly one *labeled* point.
$\mathcal{C}_1 = \mathbf{C}_1^{-1}\mathbf{Q} + \mathbf{I}$ and $\mathcal{C}_2 = \mathbf{C}_2^{-1}\mathbf{Q} + \mathbf{I}$

What is the maximal difference in the solutions?

$$\begin{aligned}
\mathbf{f}_2^\star - \mathbf{f}_1^\star &= \mathcal{C}_2^{-1}\mathbf{y}_2 - \mathcal{C}_1^{-1}\mathbf{y}_1 \\
&= \mathcal{C}_2^{-1}(\mathbf{y}_2 - \mathbf{y}_1) - \left(\mathcal{C}_1^{-1} - \mathcal{C}_2^{-1}\right)\mathbf{y}_1 \\
&= \mathcal{C}_2^{-1}(\mathbf{y}_2 - \mathbf{y}_1) - \left(\mathcal{C}_1^{-1}\left[\left(\mathbf{C}_1^{-1} - \mathbf{C}_2^{-1}\right)\mathbf{Q}\right]\mathcal{C}_2^{-1}\right)\mathbf{y}_1
\end{aligned}$$

Note that $\mathbf{v} \in \mathbb{R}^{N\times 1}$, $\lambda_m(A)\|\mathbf{v}\|_2 \le \|A\mathbf{v}\|_2 \le \lambda_M(A)\|\mathbf{v}\|_2$

$$\|\mathbf{f}_2^\star - \mathbf{f}_1^\star\|_2 \le \frac{\|\mathbf{y}_2 - \mathbf{y}_1\|_2}{\lambda_m(\mathcal{C}_2)} + \lambda_M(\mathbf{Q})$$

SSL with Graphs: Stability Bounds

$$\mathbf{f}^\star = \min_{\mathbf{f}\in\mathbb{R}^N} (\mathbf{f}-\mathbf{y})^\mathsf{T}\mathbf{C}(\mathbf{f}-\mathbf{y}) + \mathbf{f}^\mathsf{T}\mathbf{Q}\mathbf{f}$$

Think about **stability** of this solution.
Consider two datasets differing in exactly one *labeled* point.
$\mathcal{C}_1 = \mathbf{C}_1^{-1}\mathbf{Q} + \mathbf{I}$ and $\mathcal{C}_2 = \mathbf{C}_2^{-1}\mathbf{Q} + \mathbf{I}$

What is the maximal difference in the solutions?

$$\begin{aligned}
\mathbf{f}_2^\star - \mathbf{f}_1^\star &= \mathcal{C}_2^{-1}\mathbf{y}_2 - \mathcal{C}_1^{-1}\mathbf{y}_1 \\
&= \mathcal{C}_2^{-1}(\mathbf{y}_2 - \mathbf{y}_1) - \left(\mathcal{C}_1^{-1} - \mathcal{C}_2^{-1}\right)\mathbf{y}_1 \\
&= \mathcal{C}_2^{-1}(\mathbf{y}_2 - \mathbf{y}_1) - \left(\mathcal{C}_1^{-1}\left[\left(\mathbf{C}_1^{-1} - \mathbf{C}_2^{-1}\right)\mathbf{Q}\right]\mathcal{C}_2^{-1}\right)\mathbf{y}_1
\end{aligned}$$

Note that $\mathbf{v} \in \mathbb{R}^{N\times 1}$, $\lambda_m(A)\|\mathbf{v}\|_2 \le \|A\mathbf{v}\|_2 \le \lambda_M(A)\|\mathbf{v}\|_2$

$$\|\mathbf{f}_2^\star - \mathbf{f}_1^\star\|_2 \le \frac{\|\mathbf{y}_2 - \mathbf{y}_1\|_2}{\lambda_m(\mathcal{C}_2)} + \lambda_M(\mathbf{Q})\|\mathbf{C}_1^{-1} - \mathbf{C}_2^{-1}\|_2$$

SSL with Graphs: Stability Bounds

$$\mathbf{f}^\star = \min_{\mathbf{f}\in\mathbb{R}^N} (\mathbf{f}-\mathbf{y})^\mathsf{T}\mathbf{C}(\mathbf{f}-\mathbf{y}) + \mathbf{f}^\mathsf{T}\mathbf{Q}\mathbf{f}$$

Think about **stability** of this solution.

Consider two datasets differing in exactly one *labeled* point.

$\mathcal{C}_1 = \mathbf{C}_1^{-1}\mathbf{Q} + \mathbf{I}$ and $\mathcal{C}_2 = \mathbf{C}_2^{-1}\mathbf{Q} + \mathbf{I}$

What is the maximal difference in the solutions?

$$\begin{aligned}
\mathbf{f}_2^\star - \mathbf{f}_1^\star &= \mathcal{C}_2^{-1}\mathbf{y}_2 - \mathcal{C}_1^{-1}\mathbf{y}_1 \\
&= \mathcal{C}_2^{-1}(\mathbf{y}_2 - \mathbf{y}_1) - \left(\mathcal{C}_1^{-1} - \mathcal{C}_2^{-1}\right)\mathbf{y}_1 \\
&= \mathcal{C}_2^{-1}(\mathbf{y}_2 - \mathbf{y}_1) - \left(\mathcal{C}_1^{-1}\left[\left(\mathbf{C}_1^{-1} - \mathbf{C}_2^{-1}\right)\mathbf{Q}\right]\mathcal{C}_2^{-1}\right)\mathbf{y}_1
\end{aligned}$$

Note that $\mathbf{v} \in \mathbb{R}^{N\times 1}$, $\lambda_m(A)\|\mathbf{v}\|_2 \le \|A\mathbf{v}\|_2 \le \lambda_M(A)\|\mathbf{v}\|_2$

$$\|\mathbf{f}_2^\star - \mathbf{f}_1^\star\|_2 \le \frac{\|\mathbf{y}_2 - \mathbf{y}_1\|_2}{\lambda_m(\mathcal{C}_2)} + \frac{\lambda_M(\mathbf{Q})\|\mathbf{C}_1^{-1} - \mathbf{C}_2^{-1}\|_2 \cdot \|\mathbf{y}_1\|_2}{}$$

SSL with Graphs: Stability Bounds

$$\mathbf{f}^\star = \min_{\mathbf{f}\in\mathbb{R}^N} (\mathbf{f}-\mathbf{y})^\mathsf{T}\mathbf{C}(\mathbf{f}-\mathbf{y}) + \mathbf{f}^\mathsf{T}\mathbf{Q}\mathbf{f}$$

Think about **stability** of this solution.

Consider two datasets differing in exactly one *labeled* point.

$\mathcal{C}_1 = \mathbf{C}_1^{-1}\mathbf{Q} + \mathbf{I}$ and $\mathcal{C}_2 = \mathbf{C}_2^{-1}\mathbf{Q} + \mathbf{I}$

What is the maximal difference in the solutions?

$$\begin{aligned}
\mathbf{f}_2^\star - \mathbf{f}_1^\star &= \mathcal{C}_2^{-1}\mathbf{y}_2 - \mathcal{C}_1^{-1}\mathbf{y}_1 \\
&= \mathcal{C}_2^{-1}(\mathbf{y}_2 - \mathbf{y}_1) - \left(\mathcal{C}_1^{-1} - \mathcal{C}_2^{-1}\right)\mathbf{y}_1 \\
&= \mathcal{C}_2^{-1}(\mathbf{y}_2 - \mathbf{y}_1) - \left(\mathcal{C}_1^{-1}\left[\left(\mathbf{C}_1^{-1} - \mathbf{C}_2^{-1}\right)\mathbf{Q}\right]\mathcal{C}_2^{-1}\right)\mathbf{y}_1
\end{aligned}$$

Note that $\mathbf{v} \in \mathbb{R}^{N\times 1}$, $\lambda_m(A)\|\mathbf{v}\|_2 \le \|A\mathbf{v}\|_2 \le \lambda_M(A)\|\mathbf{v}\|_2$

$$\|\mathbf{f}_2^\star - \mathbf{f}_1^\star\|_2 \le \frac{\|\mathbf{y}_2 - \mathbf{y}_1\|_2}{\lambda_m(\mathcal{C}_2)} + \frac{\lambda_M(\mathbf{Q})\|\mathbf{C}_1^{-1} - \mathbf{C}_2^{-1}\|_2 \cdot \|\mathbf{y}_1\|_2}{\lambda_m(\mathcal{C}_2)\lambda_m(\mathcal{C}_1)}$$

SSL with Graphs: Stability Bounds

$$\mathbf{f}^\star = \min_{\mathbf{f}\in\mathbb{R}^N} (\mathbf{f}-\mathbf{y})^\mathsf{T}\mathbf{C}(\mathbf{f}-\mathbf{y}) + \mathbf{f}^\mathsf{T}\mathbf{Q}\mathbf{f}$$

$$\|\mathbf{f}_2^\star - \mathbf{f}_1^\star\|_2 \leq \frac{\|\mathbf{y}_2-\mathbf{y}_1\|_2}{\lambda_m(\mathcal{C}_2)} + \frac{\lambda_M(\mathbf{Q})\|\mathbf{C}_1^{-1}-\mathbf{C}_2^{-1}\|_2\cdot\|\mathbf{y}_1\|_2}{\lambda_m(\mathcal{C}_2)\lambda_m(\mathcal{C}_1)}$$

SSL with Graphs: Stability Bounds

$$\mathbf{f}^\star = \min_{\mathbf{f}\in\mathbb{R}^N} (\mathbf{f}-\mathbf{y})^\mathsf{T}\mathbf{C}(\mathbf{f}-\mathbf{y}) + \mathbf{f}^\mathsf{T}\mathbf{Q}\mathbf{f}$$

$$\|\mathbf{f}_2^\star - \mathbf{f}_1^\star\|_2 \le \frac{\|\mathbf{y}_2 - \mathbf{y}_1\|_2}{\lambda_m(\mathcal{C}_2)} + \frac{\lambda_M(\mathbf{Q})\|\mathbf{C}_1^{-1} - \mathbf{C}_2^{-1}\|_2 \cdot \|\mathbf{y}_1\|_2}{\lambda_m(\mathcal{C}_2)\lambda_m(\mathcal{C}_1)}$$

Using $\lambda_m(\mathcal{C}) \ge \frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C})} + 1$

SSL with Graphs: Stability Bounds

$$\mathbf{f}^\star = \min_{\mathbf{f}\in\mathbb{R}^N} (\mathbf{f}-\mathbf{y})^\mathsf{T}\mathbf{C}(\mathbf{f}-\mathbf{y}) + \mathbf{f}^\mathsf{T}\mathbf{Q}\mathbf{f}$$

$$\|\mathbf{f}_2^\star - \mathbf{f}_1^\star\|_2 \leq \frac{\|\mathbf{y}_2 - \mathbf{y}_1\|_2}{\lambda_m(\mathcal{C}_2)} + \frac{\lambda_M(\mathbf{Q})\|\mathbf{C}_1^{-1} - \mathbf{C}_2^{-1}\|_2 \cdot \|\mathbf{y}_1\|_2}{\lambda_m(\mathcal{C}_2)\lambda_m(\mathcal{C}_1)}$$

Using $\lambda_m(\mathcal{C}) \geq \frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C})} + 1$

$$\|\mathbf{f}_2^\star - \mathbf{f}_1^\star\|_2 \leq \frac{\|\mathbf{y}_2 - \mathbf{y}_1\|_2}{\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_1)} + 1} + \frac{\lambda_M(\mathbf{Q})\|\mathbf{C}_1^{-1} - \mathbf{C}_2^{-1}\|_2 \cdot \|\mathbf{y}_1\|_2}{\left(\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_2)} + 1\right)\left(\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_1)} + 1\right)}$$

SSL with Graphs: Stability Bounds

$$\|\mathbf{f}_2^\star - \mathbf{f}_1^\star\|_\infty \leq \beta \leq \frac{\|\mathbf{y}_2 - \mathbf{y}_1\|_2}{\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_1)} + 1} + \frac{\lambda_M(\mathbf{Q})\|\mathbf{C}_1^{-1} - \mathbf{C}_2^{-1}\|_2 \cdot \|\mathbf{y}_1\|_2}{\left(\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_2)} + 1\right)\left(\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_1)} + 1\right)}$$

SSL with Graphs: Stability Bounds

$$\|\mathbf{f}_2^\star - \mathbf{f}_1^\star\|_\infty \leq \beta \leq \frac{\|\mathbf{y}_2 - \mathbf{y}_1\|_2}{\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_1)} + 1} + \frac{\lambda_M(\mathbf{Q})\|\mathbf{C}_1^{-1} - \mathbf{C}_2^{-1}\|_2 \cdot \|\mathbf{y}_1\|_2}{\left(\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_2)} + 1\right)\left(\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_1)} + 1\right)}$$

Now, let us plug in the values for our problem.

SSL with Graphs: Stability Bounds

$$\|\mathbf{f}_2^\star - \mathbf{f}_1^\star\|_\infty \leq \beta \leq \frac{\|\mathbf{y}_2 - \mathbf{y}_1\|_2}{\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_1)} + 1} + \frac{\lambda_M(\mathbf{Q})\|\mathbf{C}_1^{-1} - \mathbf{C}_2^{-1}\|_2 \cdot \|\mathbf{y}_1\|_2}{\left(\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_2)} + 1\right)\left(\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_1)} + 1\right)}$$

Now, let us plug in the values for our problem.

Take $c_l = 1$

SSL with Graphs: Stability Bounds

$$\|\mathbf{f}_2^\star - \mathbf{f}_1^\star\|_\infty \leq \beta \leq \frac{\|\mathbf{y}_2 - \mathbf{y}_1\|_2}{\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_1)} + 1} + \frac{\lambda_M(\mathbf{Q})\|\mathbf{C}_1^{-1} - \mathbf{C}_2^{-1}\|_2 \cdot \|\mathbf{y}_1\|_2}{\left(\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_2)} + 1\right)\left(\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_1)} + 1\right)}$$

Now, let us plug in the values for our problem.

Take $c_l = 1$ and $c_l > c_u$.

SSL with Graphs: Stability Bounds

$$\|\mathbf{f}_2^\star - \mathbf{f}_1^\star\|_\infty \leq \beta \leq \frac{\|\mathbf{y}_2 - \mathbf{y}_1\|_2}{\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_1)} + 1} + \frac{\lambda_M(\mathbf{Q})\|\mathbf{C}_1^{-1} - \mathbf{C}_2^{-1}\|_2 \cdot \|\mathbf{y}_1\|_2}{\left(\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_2)} + 1\right)\left(\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_1)} + 1\right)}$$

Now, let us plug in the values for our problem.

Take $c_l = 1$ and $c_l > c_u$. We have $|y_i| \leq 1$

SSL with Graphs: Stability Bounds

$$\|\mathbf{f}_2^\star - \mathbf{f}_1^\star\|_\infty \leq \beta \leq \frac{\|\mathbf{y}_2 - \mathbf{y}_1\|_2}{\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_1)} + 1} + \frac{\lambda_M(\mathbf{Q})\|\mathbf{C}_1^{-1} - \mathbf{C}_2^{-1}\|_2 \cdot \|\mathbf{y}_1\|_2}{\left(\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_2)} + 1\right)\left(\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_1)} + 1\right)}$$

Now, let us plug in the values for our problem.

Take $c_l = 1$ and $c_l > c_u$. We have $|y_i| \leq 1$ and $|f_i^\star| \leq 1$.

SSL with Graphs: Stability Bounds

$$\|\mathbf{f}_2^\star - \mathbf{f}_1^\star\|_\infty \leq \beta \leq \frac{\|\mathbf{y}_2 - \mathbf{y}_1\|_2}{\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_1)} + 1} + \frac{\lambda_M(\mathbf{Q})\|\mathbf{C}_1^{-1} - \mathbf{C}_2^{-1}\|_2 \cdot \|\mathbf{y}_1\|_2}{\left(\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_2)} + 1\right)\left(\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_1)} + 1\right)}$$

Now, let us plug in the values for our problem.

Take $c_l = 1$ and $c_l > c_u$. We have $|y_i| \leq 1$ and $|f_i^\star| \leq 1$.

$$\beta \leq 2\left[\frac{\sqrt{2}}{\lambda_m(\mathbf{Q}) + 1} + \sqrt{2n_l}\frac{1 - c_u}{c_u}\frac{\lambda_M(\mathbf{Q})}{(\lambda_m(\mathbf{Q}) + 1)^2}\right]$$

SSL with Graphs: Stability Bounds

$$\|\mathbf{f}_2^\star - \mathbf{f}_1^\star\|_\infty \leq \beta \leq \frac{\|\mathbf{y}_2 - \mathbf{y}_1\|_2}{\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_1)} + 1} + \frac{\lambda_M(\mathbf{Q})\|\mathbf{C}_1^{-1} - \mathbf{C}_2^{-1}\|_2 \cdot \|\mathbf{y}_1\|_2}{\left(\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_2)} + 1\right)\left(\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_1)} + 1\right)}$$

Now, let us plug in the values for our problem.

Take $c_l = 1$ and $c_l > c_u$. We have $|y_i| \leq 1$ and $|f_i^\star| \leq 1$.

$$\beta \leq 2\left[\frac{\sqrt{2}}{\lambda_m(\mathbf{Q}) + 1} + \sqrt{2n_l}\frac{1 - c_u}{c_u}\frac{\lambda_M(\mathbf{Q})}{(\lambda_m(\mathbf{Q}) + 1)^2}\right]$$

$\mathbf{Q}$ is reg. $\mathbf{L}$: $\lambda_m(\mathbf{Q}) = \lambda_m(\mathbf{L}) + \gamma_g$

SSL with Graphs: Stability Bounds

$$\|\mathbf{f}_2^\star - \mathbf{f}_1^\star\|_\infty \leq \beta \leq \frac{\|\mathbf{y}_2 - \mathbf{y}_1\|_2}{\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_1)} + 1} + \frac{\lambda_M(\mathbf{Q})\|\mathbf{C}_1^{-1} - \mathbf{C}_2^{-1}\|_2 \cdot \|\mathbf{y}_1\|_2}{\left(\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_2)} + 1\right)\left(\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_1)} + 1\right)}$$

Now, let us plug in the values for our problem.

Take $c_l = 1$ and $c_l > c_u$. We have $|y_i| \leq 1$ and $|f_i^\star| \leq 1$.

$$\beta \leq 2\left[\frac{\sqrt{2}}{\lambda_m(\mathbf{Q}) + 1} + \sqrt{2n_l}\frac{1 - c_u}{c_u}\frac{\lambda_M(\mathbf{Q})}{(\lambda_m(\mathbf{Q}) + 1)^2}\right]$$

$\mathbf{Q}$ is reg. $\mathbf{L}$: $\lambda_m(\mathbf{Q}) = \lambda_m(\mathbf{L}) + \gamma_g$ and $\lambda_M(\mathbf{Q}) = \lambda_M(\mathbf{L}) + \gamma_g$

SSL with Graphs: Stability Bounds

$$\|\mathbf{f}_2^\star - \mathbf{f}_1^\star\|_\infty \leq \beta \leq \frac{\|\mathbf{y}_2 - \mathbf{y}_1\|_2}{\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_1)} + 1} + \frac{\lambda_M(\mathbf{Q})\|\mathbf{C}_1^{-1} - \mathbf{C}_2^{-1}\|_2 \cdot \|\mathbf{y}_1\|_2}{\left(\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_2)} + 1\right)\left(\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_1)} + 1\right)}$$

Now, let us plug in the values for our problem.

Take $c_l = 1$ and $c_l > c_u$. We have $|y_i| \leq 1$ and $|f_i^\star| \leq 1$.

$$\beta \leq 2\left[\frac{\sqrt{2}}{\lambda_m(\mathbf{Q}) + 1} + \sqrt{2n_l}\frac{1 - c_u}{c_u}\frac{\lambda_M(\mathbf{Q})}{(\lambda_m(\mathbf{Q}) + 1)^2}\right]$$

$\mathbf{Q}$ is reg. $\mathbf{L}$: $\lambda_m(\mathbf{Q}) = \lambda_m(\mathbf{L}) + \gamma_g$ and $\lambda_M(\mathbf{Q}) = \lambda_M(\mathbf{L}) + \gamma_g$

$$\beta \;\leq\; 2\left[\frac{\sqrt{2}}{\gamma_g + 1} + \sqrt{2n_l}\frac{1 - c_u}{c_u}\frac{\lambda_M(\mathbf{L}) + \gamma_g}{\gamma_g^2 + 1}\right]$$

SSL with Graphs: Stability Bounds

$$\|\mathbf{f}_2^\star - \mathbf{f}_1^\star\|_\infty \leq \beta \leq \frac{\|\mathbf{y}_2 - \mathbf{y}_1\|_2}{\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_1)} + 1} + \frac{\lambda_M(\mathbf{Q})\|\mathbf{C}_1^{-1} - \mathbf{C}_2^{-1}\|_2 \cdot \|\mathbf{y}_1\|_2}{\left(\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_2)} + 1\right)\left(\frac{\lambda_m(\mathbf{Q})}{\lambda_M(\mathbf{C}_1)} + 1\right)}$$

Now, let us plug in the values for our problem.

Take $c_l = 1$ and $c_l > c_u$. We have $|y_i| \leq 1$ and $|f_i^\star| \leq 1$.

$$\beta \leq 2\left[\frac{\sqrt{2}}{\lambda_m(\mathbf{Q}) + 1} + \sqrt{2n_l}\frac{1 - c_u}{c_u}\frac{\lambda_M(\mathbf{Q})}{(\lambda_m(\mathbf{Q}) + 1)^2}\right]$$

$\mathbf{Q}$ is reg. $\mathbf{L}$: $\lambda_m(\mathbf{Q}) = \lambda_m(\mathbf{L}) + \gamma_g$ and $\lambda_M(\mathbf{Q}) = \lambda_M(\mathbf{L}) + \gamma_g$

$$\beta \;\leq\; 2\left[\frac{\sqrt{2}}{\gamma_g + 1} + \sqrt{2n_l}\frac{1 - c_u}{c_u}\frac{\lambda_M(\mathbf{L}) + \gamma_g}{\gamma_g^2 + 1}\right]$$

This algorithm is β-stable!

SSL with Graphs: Regularized Harmonic Functions

Larger implications of random walks

random walk relates to **commute distance**

SSL with Graphs: Regularized Harmonic Functions

Larger implications of random walks

random walk relates to **commute distance** which should satisfy

$(\star)$ Vertices in the **same** cluster of the graph have a **small** commute distance, whereas two vertices in **different** clusters of the graph have a **large** commute distance.

SSL with Graphs: Regularized Harmonic Functions

Larger implications of random walks

random walk relates to **commute distance** which should satisfy

($\star$) Vertices in the **same** cluster of the graph have a **small** commute distance, whereas two vertices in **different** clusters of the graph have a **large** commute distance.

Do we have this property for HS?

SSL with Graphs: Regularized Harmonic Functions

Larger implications of random walks

random walk relates to **commute distance** which should satisfy

($\star$) Vertices in the **same** cluster of the graph have a **small** commute distance, whereas two vertices in **different** clusters of the graph have a **large** commute distance.

Do we have this property for HS? What if $N \to \infty$?

SSL with Graphs: Regularized Harmonic Functions

Larger implications of random walks

random walk relates to **commute distance** which should satisfy

($\star$) Vertices in the **same** cluster of the graph have a **small** commute distance, whereas two vertices in **different** clusters of the graph have a **large** commute distance.

Do we have this property for HS? What if $N \to \infty$?

Luxburg/Radl/Hein: *Getting lost in space: Large sample analysis of the commute distance*

`http://www.informatik.uni-hamburg.de/ML/contents/people/luxburg/publications/LuxburgRadlHein2010_PaperAndSupplement.pdf`

SSL with Graphs: Regularized Harmonic Functions

Larger implications of random walks

random walk relates to **commute distance** which should satisfy

($\star$) Vertices in the **same** cluster of the graph have a **small** commute distance, whereas two vertices in **different** clusters of the graph have a **large** commute distance.

Do we have this property for HS? What if $N \to \infty$?

Luxburg/Radl/Hein: *Getting lost in space: Large sample analysis of the commute distance*

`http://www.informatik.uni-hamburg.de/ML/contents/people/luxburg/publications/LuxburgRadlHein2010_PaperAndSupplement.pdf`

Solutions?

SSL with Graphs: Regularized Harmonic Functions

Larger implications of random walks

random walk relates to **commute distance** which should satisfy

($\star$) Vertices in the **same** cluster of the graph have a **small** commute distance, whereas two vertices in **different** clusters of the graph have a **large** commute distance.

Do we have this property for HS? What if $N \to \infty$?

Luxburg/Radl/Hein: *Getting lost in space: Large sample analysis of the commute distance*

`http://www.informatik.uni-hamburg.de/ML/contents/people/luxburg/publications/LuxburgRadlHein2010_PaperAndSupplement.pdf`

Solutions? 1) γ_g

SSL with Graphs: Regularized Harmonic Functions

Larger implications of random walks

random walk relates to **commute distance** which should satisfy

($\star$) Vertices in the **same** cluster of the graph have a **small** commute distance, whereas two vertices in **different** clusters of the graph have a **large** commute distance.

Do we have this property for HS? What if $N \to \infty$?

Luxburg/Radl/Hein: *Getting lost in space: Large sample analysis of the commute distance*

`http://www.informatik.uni-hamburg.de/ML/contents/people/luxburg/publications/LuxburgRadlHein2010_PaperAndSupplement.pdf`

Solutions? 1) γ_g 2) amplified commute distance

SSL with Graphs: Regularized Harmonic Functions

Larger implications of random walks

random walk relates to **commute distance** which should satisfy

($\star$) Vertices in the **same** cluster of the graph have a **small** commute distance, whereas two vertices in **different** clusters of the graph have a **large** commute distance.

Do we have this property for HS? What if $N \to \infty$?

Luxburg/Radl/Hein: *Getting lost in space: Large sample analysis of the commute distance*

http://www.informatik.uni-hamburg.de/ML/contents/people/luxburg/publications/LuxburgRadlHein2010_PaperAndSupplement.pdf

Solutions? 1) γ_g 2) amplified commute distance 3) $\mathbf{L}^p$

SSL with Graphs: Regularized Harmonic Functions

Larger implications of random walks

random walk relates to **commute distance** which should satisfy

($\star$) Vertices in the **same** cluster of the graph have a **small** commute distance, whereas two vertices in **different** clusters of the graph have a **large** commute distance.

Do we have this property for HS? What if $N \to \infty$?

Luxburg/Radl/Hein: *Getting lost in space: Large sample analysis of the commute distance*

http://www.informatik.uni-hamburg.de/ML/contents/people/luxburg/publications/LuxburgRadlHein2010_PaperAndSupplement.pdf

Solutions? 1) γ_g 2) amplified commute distance 3) $\mathbf{L}^p$ ...

SSL with Graphs: Regularized Harmonic Functions

Larger implications of random walks

random walk relates to **commute distance** which should satisfy

($\star$) Vertices in the **same** cluster of the graph have a **small** commute distance, whereas two vertices in **different** clusters of the graph have a **large** commute distance.

Do we have this property for HS? What if $N \to \infty$?

Luxburg/Radl/Hein: *Getting lost in space: Large sample analysis of the commute distance*

`http://www.informatik.uni-hamburg.de/ML/contents/people/luxburg/publications/LuxburgRadlHein2010_PaperAndSupplement.pdf`

Solutions? 1) γ_g 2) amplified commute distance 3) $\mathbf{L}^p$ ...

SSL with Graphs: Regularized Harmonic Functions

Larger implications of random walks

random walk relates to **commute distance** which should satisfy

(⋆) Vertices in the **same** cluster of the graph have a **small** commute distance, whereas two vertices in **different** clusters of the graph have a **large** commute distance.

Do we have this property for HS? What if $N \to \infty$?

Luxburg/Radl/Hein: *Getting lost in space: Large sample analysis of the commute distance*

`http://www.informatik.uni-hamburg.de/ML/contents/people/luxburg/publications/LuxburgRadlHein2010_PaperAndSupplement.pdf`

Solutions? 1) γ_g 2) amplified commute distance 3) $\mathbf{L}^p$ ...

SSL with Graphs: Out of sample extension

Both **MinCut** and **HFS** only inferred the labels on unlabeled data.

They are

SSL with Graphs: Out of sample extension

Both **MinCut** and **HFS** only inferred the labels on unlabeled data.

They are **transductive**.

SSL with Graphs: Out of sample extension

Both **MinCut** and **HFS** only inferred the labels on unlabeled data.

They are **transductive**.

What if a new point $\mathbf{x}_{n_l+n_u+1}$ arrives?

SSL with Graphs: Out of sample extension

Both **MinCut** and **HFS** only inferred the labels on unlabeled data.

They are **transductive**.

What if a new point $\mathbf{x}_{n_l+n_u+1}$ arrives? also called out-of-sample extension

SSL with Graphs: Out of sample extension

Both **MinCut** and **HFS** only inferred the labels on unlabeled data.

They are **transductive**.

What if a new point $\mathbf{x}_{n_l+n_u+1}$ arrives? also called out-of-sample extension

Option 1) Add it to the graph and recompute HFS.

SSL with Graphs: Out of sample extension

Both **MinCut** and **HFS** only inferred the labels on unlabeled data.

They are **transductive**.

What if a new point $\mathbf{x}_{n_l+n_u+1}$ arrives? also called out-of-sample extension

Option 1) Add it to the graph and recompute HFS.

Option 2) Make the algorithms **inductive**!

SSL with Graphs: Out of sample extension

Both **MinCut** and **HFS** only inferred the labels on unlabeled data.

They are **transductive**.

What if a new point $\mathbf{x}_{n_l+n_u+1}$ arrives? also called out-of-sample extension

Option 1) Add it to the graph and recompute HFS.

Option 2) Make the algorithms **inductive**!

Allow to be defined everywhere: $f : \mathcal{X} \mapsto \mathbb{R}$

SSL with Graphs: Out of sample extension

Both **MinCut** and **HFS** only inferred the labels on unlabeled data.

They are **transductive**.

What if a new point $\mathbf{x}_{n_l+n_u+1}$ arrives? also called out-of-sample extension

Option 1) Add it to the graph and recompute HFS.

Option 2) Make the algorithms **inductive**!

Allow to be defined everywhere: $f : \mathcal{X} \mapsto \mathbb{R}$
Allow $f(\mathbf{x}_i) \neq y_i$.

SSL with Graphs: Out of sample extension

Both **MinCut** and **HFS** only inferred the labels on unlabeled data.

They are **transductive**.

What if a new point $\mathbf{x}_{n_l+n_u+1}$ arrives? also called out-of-sample extension

Option 1) Add it to the graph and recompute HFS.

Option 2) Make the algorithms **inductive**!

Allow to be defined everywhere: $f : \mathcal{X} \mapsto \mathbb{R}$
Allow $f(\mathbf{x}_i) \neq y_i$. Why?

SSL with Graphs: Out of sample extension

Both **MinCut** and **HFS** only inferred the labels on unlabeled data.

They are **transductive**.

What if a new point $\mathbf{x}_{n_l+n_u+1}$ arrives? also called out-of-sample extension

Option 1) Add it to the graph and recompute HFS.

Option 2) Make the algorithms **inductive**!

Allow to be defined everywhere: $f : \mathcal{X} \mapsto \mathbb{R}$
Allow $f(\mathbf{x}_i) \neq y_i$. Why? To deal with noise.

SSL with Graphs: Out of sample extension

Both **MinCut** and **HFS** only inferred the labels on unlabeled data.

They are **transductive**.

What if a new point $\mathbf{x}_{n_l+n_u+1}$ arrives? also called out-of-sample extension

Option 1) Add it to the graph and recompute HFS.

Option 2) Make the algorithms **inductive**!

Allow to be defined everywhere: $f : \mathcal{X} \mapsto \mathbb{R}$
Allow $f(\mathbf{x}_i) \neq y_i$. Why? To deal with noise.

Solution: **Manifold Regularization**

Michal Valko

michal.valko@inria.fr

Inria & ENS Paris-Saclay, MVA

https://misovalko.github.io/mva-ml-graphs.html