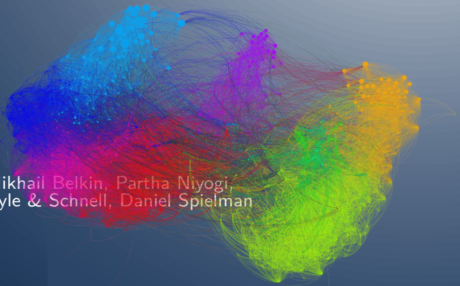
# Graphs in Machine Learning

## Manifold Learning

### Laplacian Eigenmaps


Michal Valko

*Inria & ENS Paris-Saclay, MVA*

Partially based on material by: Ulrike von Luxburg, Mikhail Belkin, Partha Niyogi, Olivier Chapelle, Bernhard Schölkopf, Gary Miller, Doyle & Schnell, Daniel Spielman

# Background: Manifold Learning

**problem: definition reduction/manifold learning**

Given $\{\mathbf{x}_i\}_{i=1}^{N}$ from $\mathbb{R}^d$ find $\{\mathbf{y}_i\}_{i=1}^{N}$ in $\mathbb{R}^m$, where $m \ll d$.

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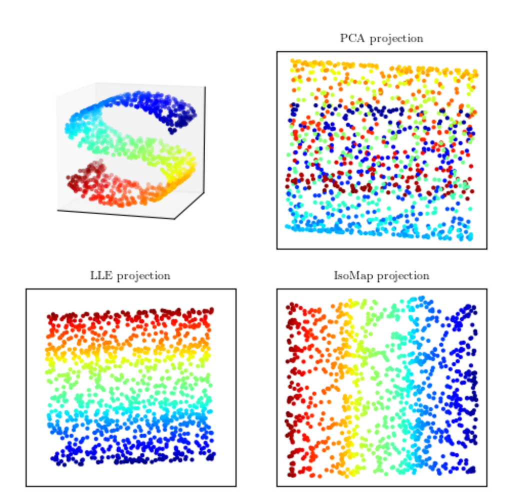
  - nonlinear often preserve only **local** distances

# Manifold Learning: Linear vs. Non-linear

# Manifold Learning: Linear vs. Non-linear (Alternative View)

What do we know about linear vs. nonlinear methods?



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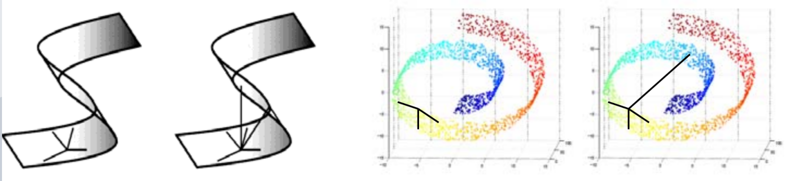

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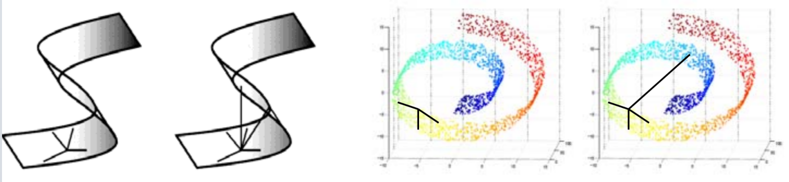

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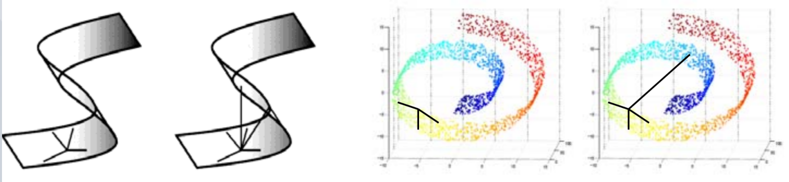

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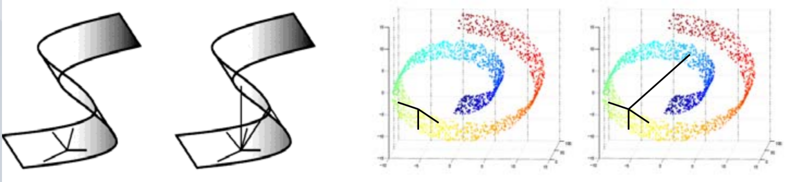

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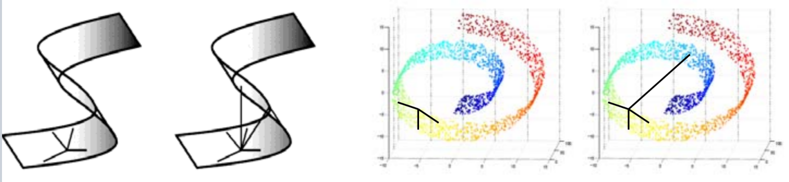

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Looks familiar?

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**Note$_2$:** $\mathbf{f}_1$ is useless

`http://web.cse.ohio-state.edu/~mbelkin/papers/LEM_NC_03.pdf`

# Manifold Learning: Laplacian Eigenmaps to 1D

### Laplacian Eigenmaps 1D objective

$$\min_{\mathbf{f}} \mathbf{f}^\mathsf{T}\mathbf{L}\mathbf{f} \quad \text{s.t.} \quad f_i \in \mathbb{R}, \quad \mathbf{f}^\mathsf{T}\mathbf{D}\mathbf{1} = 0, \quad \mathbf{f}^\mathsf{T}\mathbf{D}\mathbf{f} = \mathbf{1}$$

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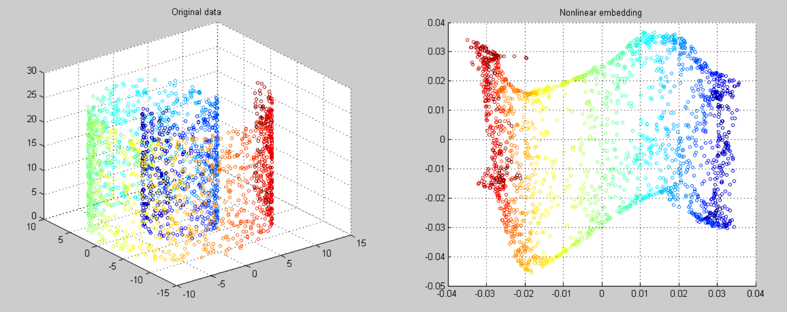
What is the solution?

# Manifold Learning: Example




`http://www.mathworks.com/matlabcentral/fileexchange/`

`36141-laplacian-eigenmap-~-diffusion-map-~-manifold-learning`

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https://misovalko.github.io/mva-ml-graphs.html