# Graphs in Machine Learning

## Spectral Clustering: Relaxation

From Discrete to Continuous


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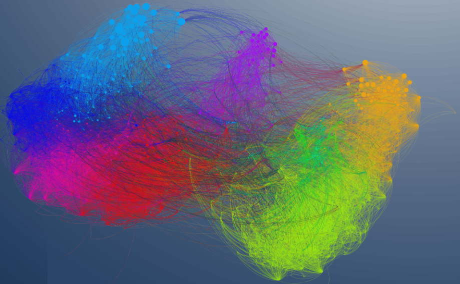
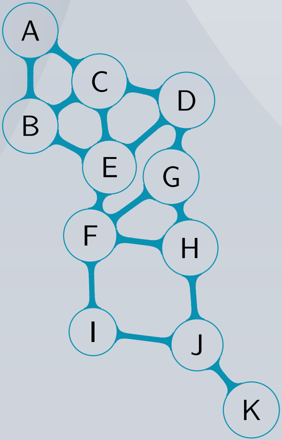
Partially based on material by: Ulrike von Luxburg, Gary Miller, Doyle & Schnell, Daniel Spielman



# Spectral Clustering: Cuts on graphs



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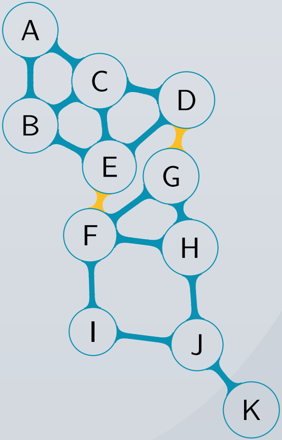


Defining the cut objective we get the clustering!

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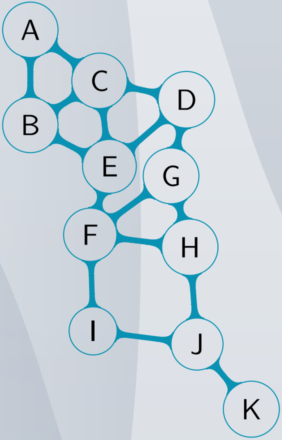


**MinCut**: $\mathrm{cut}(A, B) = \sum_{i \in A, j \in B} w_{ij}$

Are we done?

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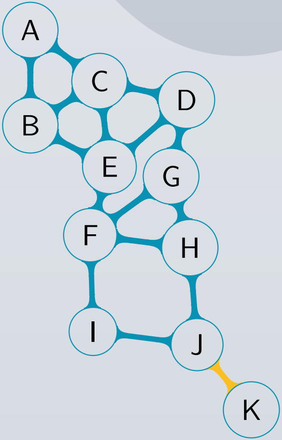


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Can be solved efficiently, but maybe not what we want . . . .

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Approximate!

# Spectral Clustering: Relaxing Balanced Cuts

## Relaxation for (simple) balanced cuts for 2 sets

$$\min_{A,B} \operatorname{cut}(A,B) \quad \text{s.t.} \quad |A| = |B|$$

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What is the relationship with the **smoothness** of a graph function?

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**objective function of spectral clustering**

$$\min_{\mathbf{f}} \mathbf{f}^\mathsf{T}\mathbf{L}\mathbf{f} \quad \text{s.t.} \quad f_i = \pm 1, \quad \mathbf{f} \perp \mathbf{1}_N, \quad \|\mathbf{f}\| = \sqrt{N}$$

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$$\cancel{f_i = \pm 1} \quad \rightarrow \quad f_i \in \mathbb{R}$$

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**Rayleigh-Ritz theorem**

If $\lambda_1 \leq \cdots \leq \lambda_N$ are the eigenvalues of real symmetric $\mathbf{L}$ then

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$\frac{\mathbf{x}^\mathsf{T}\mathbf{L}\mathbf{x}}{\mathbf{x}^\mathsf{T}\mathbf{x}} \equiv$ **Rayleigh quotient**

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How can we use it?

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**Generalized Rayleigh-Ritz theorem (Courant-Fischer-Weyl)**

If $\lambda_1 \leq \cdots \leq \lambda_N$ are the eigenvalues of real symmetric $\mathbf{L}$ and $\mathbf{v}_1, \ldots, \mathbf{v}_N$ the corresponding orthogonal eigenvectors, then for $k = 1 : N - 1$

$$\lambda_{k+1} = \min_{\mathbf{x} \neq 0, \mathbf{x} \perp \mathbf{v}_1, \ldots \mathbf{v}_k} \frac{\mathbf{x}^\mathsf{T}\mathbf{L}\mathbf{x}}{\mathbf{x}^\mathsf{T}\mathbf{x}} = \min_{\mathbf{x}^\mathsf{T}\mathbf{x} = 1, \mathbf{x} \perp \mathbf{v}_1, \ldots \mathbf{v}_k} \mathbf{x}^\mathsf{T}\mathbf{L}\mathbf{x}$$

$$\lambda_{N-k} = \max_{\mathbf{x} \neq 0, \mathbf{x} \perp \mathbf{v}_n, \ldots \mathbf{v}_{N-k+1}} \frac{\mathbf{x}^\mathsf{T}\mathbf{L}\mathbf{x}}{\mathbf{x}^\mathsf{T}\mathbf{x}} = \max_{\mathbf{x}^\mathsf{T}\mathbf{x} = 1, \mathbf{x} \perp \mathbf{v}_N, \ldots \mathbf{v}_{N-k+1}} \mathbf{x}^\mathsf{T}\mathbf{L}\mathbf{x}$$

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https://misovalko.github.io/mva-ml-graphs.html