Graphs in Machine Learning

Spectral Clustering: NCut

Normalized Cut Approximation

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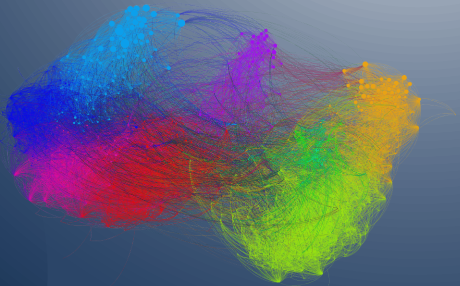
Partially based on material by: Ulrike von Luxburg, Gary Miller, Doyle & Schnell, Daniel Spielman



Spectral Clustering: Approximating NCut

Normalized Cut

$$\mathrm{NCut}(A, B) = \sum_{i \in A, j \in B} w_{ij} \left(\frac{1}{\mathrm{vol}(A)} + \frac{1}{\mathrm{vol}(B)} \right)$$

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Define graph function $\mathbf{f}$ for cluster membership of NCut:

$$f_i = \begin{cases} \sqrt{\frac{\mathrm{vol}(B)}{\mathrm{vol}(A)}} & \text{if } V_i \in A, \\ -\sqrt{\frac{\mathrm{vol}(A)}{\mathrm{vol}(B)}} & \text{if } V_i \in B. \end{cases}$$

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objective function of spectral clustering (NCut)

$$\min_{\mathbf{f}} \mathbf{f}^\mathsf{T}\mathbf{Lf} \quad \text{s.t.} \quad f_i \in \mathbb{R}, \quad \mathbf{Df} \perp \mathbf{1}_N, \quad \mathbf{f}^\mathsf{T}\mathbf{Df} = \mathrm{vol}(\mathcal{V})$$

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tl;dr: Get the second eigenvector of $\mathbf{L}/\mathbf{L}_{\text{rw}}$ for RatioCut/NCut.

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demo: `https://dominikschmidt.xyz/spectral-clustering-exp/`

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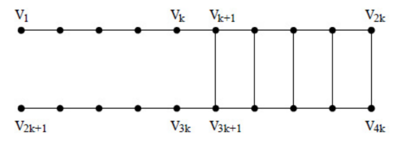
Pretty bad. Example: cockroach graphs



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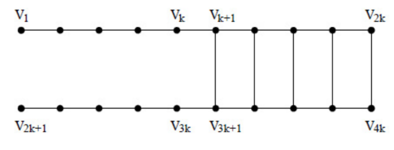

No efficient approximation exist. Other relaxations possible.

https://www.cs.cmu.edu/~glmiller/Publications/Papers/GuMi95.pdf

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https://misovalko.github.io/mva-ml-graphs.html