Graphs in Machine Learning

Laplacian and Random Walks

Stationary Distribution

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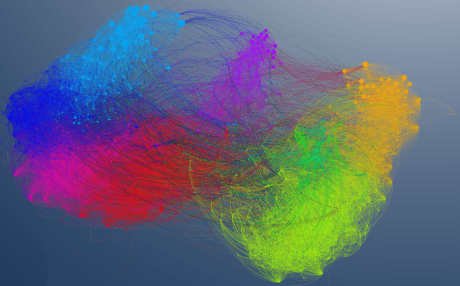
Partially based on material by: Ulrike von Luxburg, Gary Miller, Doyle & Schnell, Daniel Spielman



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https://misovalko.github.io/mva-ml-graphs.html